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"Research Note"

ANALYSIS OF THREE-DIMENSIONAL CONSOLIDATION OF UNSATURATED SOILS*

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Abstract– This study extends the theory of three-dimensional consolidation to unsaturated soils and formulates the theory for finite element analysis by treating the pore water and pore air as a mixed pore fluid. This formulation considers variations in the permeability and compressibility of the mixed pore fluid with changes in the void ratio and degree of saturation. The compressibility of the mixed pore fluid is derived using Boyle's Law. An example of the settlement of a vertical drain is investigated and discussed; this example demonstrates that the numerical analysis theory is applicable and reliable. The results indicate that the rate of consolidation of unsaturated soils is clearly slower than that of saturated soils, the rate of dissipation of the pore fluid pressure is considerably slower, and the permeability of the mixed pore fluid decrease during consolidation. This theory is applicable to unsaturated soils with high degrees of saturation and can be used to obtain more reliable predictions of unsaturated soil consolidation.

Keywords– Unsaturated soils, consolidation theory, pore fluid, finite element, vertical drain

1. INTRODUCTION

The consolidation of soils as a result of the dissipation of excess pore pressures has been of considerable concern in both practice and theory in soil mechanics for many years. Biot's theory of consolidation [1] is preferable from a theoretical perspective because it incorporates the coupling effect between the dissipation of excess pore water pressure and deformation of the soil skeleton during consolidation. However, Biot's theory assumes complete soil saturation, whereas many practical problems involve the consolidation of unsaturated compacted soils, such as the filling of earth dams, roadways, and railways with compacted fill. Blight [2] derived a consolidation equation for the pore air in dry and hard unsaturated soil using Fick's law. Scott [3] introduced changes in the terms for the void ratio and degree of saturation in the consolidation equation for unsaturated soils containing closed air bubbles. Barden [4] first proposed a consolidation model that considered the coupling of deformation, pore water pressure, and pore air pressure. Fredlund and Hasan [5-7] proposed the use of two partial differential equations to solve for the pore water pressure and pore air pressure, assuming that the air phase is continuous. Dakshanamurthy [8] established a numerical model to describe the pore water–air flow based on Fredlund and Hasan's theory. Ausilio [9] investigated the one-dimensional consolidation of unsaturated soil based on increasing load or matric suction. Fazeli [10] conducted ten triaxial tests which were carried out to study the unsaturated shear strength characteristics of Shiraz silty clay soil. QIN Aifang [11-12] derived semi-analytical and analytical solutions based on Fredlund's one-dimensional consolidation theory for unsaturated soils. Badv [13] conducted a series of consolidation and direct shear tests and investigated the relationship between the key mechanical and physical properties.

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The finite element method has expanded the range of applicability of consolidation theory for unsaturated soils. Chang and Duncan [14] extended the theory of consolidation to partially saturated clay soils and developed a finite element formulation for the problem. Narasimhan [15] presented a method for numerically simulating the movement of water in variably saturated deformable porous media that considers a general three-dimensional field of flow in conjunction with a one-dimensional vertical deformation field. Schrefler [16-17] developed a fully coupled model to simulate the slow transient phenomena involved in the flow of water and air in deformable porous media. The finite element method is used for the discrete approximation of the partial differential equations governing the problem. Wong Tai [18] investigated a coupled numerical simulation of the consolidation of unsaturated soils. Conte [19] solved simplified consolidation equations for unsaturated soils based on the consolidation theory proposed by Fredlund and his coworkers using a Fourier transform. This paper presents an extension of Biot’s theory that permits finite element analyses of the consolidation of unsaturated soils with arbitrary geometry and boundary conditions. The theory considers variations in the permeability and compressibility of the mixed pore fluid with changes in the void ratio and degree of saturation. This theory is applicable to unsaturated soils with high degrees of saturation and can be used to obtain more reliable predictions of unsaturated soil consolidation.

2. CONTROL EQUATIONS OF CONSOLIDATION

For unsaturated soils with high degrees of saturation, the pore air is sealed in the pore water, and thus, consolidation equations can be established by considering the pore water and air to be a mixed pore fluid. We can then establish a simplified mixed pore fluid continuity equation.

(1) Pore fluid pressure

Because the pore air is sealed in the pore water, Hanbing Bian [20] assumed that the pore water pressure u_w is equal to the pore air pressure u_a. Then, the mixed pore fluid pressure u_m can be expressed as

$$u_m = u_w \tag{1}$$

Under this condition, the effective stress principle formula can be expressed in the same form as Terzaghi's effective stress principle for saturated soil, which states that the total stress is borne by the skeleton stress and mixed pore fluid pressure:

$$\{\sigma\} = \{\sigma^{'}\} + \{M\}u_m \tag{2}$$

(2) Equilibrium differential equations

The equilibrium differential equations can be written as

$$[\partial]^T\{\sigma^{'}\} + [\partial]^T\{M\}u_m = \{f\} \tag{3}$$

Substituting the constitutive equations and geometric equations, we can obtain the equilibrium differential equations expressed in terms of displacement and pore fluid pressure:

$$-[\partial]^T[D][\partial]\{\omega\} + [\partial]^T\{M\}u_m = \{f\} \tag{4}$$

where $[\partial]^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}$, $\{M\} = [1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0]^T$, $[D]$ is the stiffness matrix, and $\{f\} = [f_x \quad f_y \quad f_z]$ is the volume force in three directions.

(3) Continuity equation of pore fluid

The unit volume compression ($\frac{\partial \varepsilon_v}{\partial t}$) is equal to the volume of fluid discharged from a unit volume and the residual pore fluid compression ($\frac{\partial \varepsilon_{v1}}{\partial t}$).

$$\frac{\partial \varepsilon_v}{\partial t} = -\frac{1}{r_m}\frac{\partial}{\partial x}(K_m \frac{\partial u_m}{\partial x}) - \frac{1}{r_m}\frac{\partial}{\partial y}(K_m \frac{\partial u_m}{\partial y}) - \frac{1}{r_m}\frac{\partial}{\partial z}(K_m \frac{\partial u_m}{\partial z}) + \frac{\partial \varepsilon_{v1}}{\partial t} \tag{5a}$$

The nonlinear finite element formulation presented in this paper uses an incremental method. Assuming that the pore fluid density r_m and permeability coefficient K_m are constant during the time interval Δt , Eq. (5a) can be expressed as

$$\frac{\partial \varepsilon_v}{\partial t} = -\frac{K_m}{r_m}\nabla^2 u_m + \frac{\partial \varepsilon_{v1}}{\partial t} \tag{5b}$$

The residual pore fluid compression is caused by changing pore fluid pressure. Therefore,

$$\frac{\partial \varepsilon_{v1}}{\partial t} = \frac{1}{B_m}\frac{\partial u_m}{\partial t} \tag{6}$$

where $B_m = -\frac{du_m}{d\varepsilon_{v1}}$ is the volume modulus of compression and $d\varepsilon_{v1}$ is the volumetric strain in the pore fluid due to changes (du_m) in the fluid pressure.

Substituting Eq. (6) into the continuity Eq. (5b), the continuity equation can be expressed as

$$\{M\}^T \frac{\partial}{\partial t}[\partial]\{\omega\} - \frac{K_m}{r_m}\nabla^2 u_m + \frac{1}{B_m}\frac{\partial u_m}{\partial t} = 0 \tag{7}$$

Finally, we obtain simultaneous differential equations that can be solved to obtain the displacement and mixed pore fluid pressure u_m:

$$\begin{cases} -[\partial]^T[D][\partial]\{\omega\} + [\partial]^T\{M\}u_m = \{f\} \\ \{M\}^T \frac{\partial}{\partial t}[\partial]\{\omega\} - \frac{K_m}{r_m}\nabla^2 u_m + \frac{1}{B_m}\frac{\partial u_m}{\partial t} = 0 \end{cases} \tag{8}$$

3. FINITE ELEMENT EQUATIONS OF CONSOLIDATION

(1) Equilibrium differential equations

The finite element formulation of the differential equilibrium equations can be established using the weighted residual method as follows:

$$\iint_e [B]^T\{\sigma^{'}\}dxdydz + \iint_e [B]^T\{M\}u_m dxdydz = \{F\}^e \tag{9}$$

Because $\{\omega\} = [N]\{\delta\}^e$ and $u_m = [N^{'}]\{p_m\}^e$, the equations can be written as

$$\iint_e [B]^T[D][B]dxdydz\{\delta\}^e + \iint_e [B]^T\{M\}[N^{'}]dxdydz\{p_m\}^e = \{F\}^e \tag{10}$$

where $[N]=\begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & & 0 & N_8 \end{bmatrix}$, $[N^{'}]=[N_1 \quad N_2 \quad \ldots \quad N_8]$, $\{\delta\}^e$ is the unit nodal displacement vector, $\{p_m\}^e$ is the unit nodal pore fluid pressure vector, and $\{F\}^e$ is the element nodal force. Assume that $[\bar{k}]=\iiint_e [B]^T[D][B]dxdydz$ and $[k^{'}]=\iiint_e [B]^T\{M\}[N^{'}]dxdydz$.

Then, Eq. (10) can be expressed as

$$[\bar{k}]\{\delta\}^e+[k^{'}]\{p_m\}^e=\{F\}^e \tag{11}$$

We can then obtain the general equilibrium equations by using the increment of displacement

$$[\bar{K}]\{\Delta\delta\}+[K^{'}]\{p_m\}=\{\Delta R\}+[K^{'}]\{p_{m,t-1}\} \tag{12}$$

where $\{\Delta R\}$ is the load increment and $\{p_{m,t-1}\}$ is the pore fluid pressure at the previous time step.

(2) Continuity equation of the pore fluid

Using the difference method and increment of displacement, the continuity equation can be expressed as

$$\{M\}^T[\partial]\{\Delta\omega\}-\frac{k_m\Delta t}{r_m}\nabla^2 u_m+\frac{1}{B_m}\Delta u_m=0 \tag{13}$$

The corresponding finite element equation can be established using the principle of virtual work:

$$[k^{'}]^T\{\Delta\delta\}^e+[k_m]\{p_m\}^e-[\hat{k}]\{p_{m,t-1}\}^e=\{Q_{m1}\}^e\Delta t \tag{14}$$

where $[k^{'}]^T=\iiint_e [N^{'}]^T\{M\}^T[B]dxdydz$, $[k_m]=[\tilde{k}]+[\hat{k}]$, $[\hat{k}]=\frac{1}{B_m}\iiint_e [N^{'}]^T[N^{'}]dxdydz$,

$[\tilde{k}]=-\frac{k_m\Delta t}{r_m}\iiint_e\left(\frac{\partial[N^{'}]^T}{\partial x}\frac{\partial[N^{'}]}{\partial x}+\frac{\partial[N^{'}]^T}{\partial y}\frac{\partial[N^{'}]}{\partial y}+\frac{\partial[N^{'}]^T}{\partial z}\frac{\partial[N^{'}]}{\partial z}\right)dxdydz$, and

$\{Q_{m1}\}^e$ is the flux of the pore fluid flowing from each node in each unit.

In the grid, the sum of the flux of the pore fluid flowing from each unit around a node is zero, i.e.,

$$\sum_e Q_{m1i}=0 \tag{15}$$

Therefore, the corresponding general continuity equation can be established.

$$[K^{'}]^T\{\Delta\delta\}+[K_m]\{p_m\}=[\hat{K}]\{p_{m,t-1}\} \tag{16}$$

Finally, we obtain the simultaneous finite element equations, which can be expressed as

$$\begin{bmatrix} \bar{K} & K^{'} \\ K^{'T} & K_m \end{bmatrix}\begin{Bmatrix} \Delta\delta \\ p_m \end{Bmatrix}=\begin{Bmatrix} \Delta R+R_t \\ V_{m1} \end{Bmatrix} \tag{17}$$

where $\{R_t\}=[K^{'}]\{p_{m,t-1}\}$ and $\{V_{m1}\}=[\hat{K}]\{p_{m,t-1}\}$.

The displacement and mixed pore fluid pressure can be obtained using Eq. (17). This method is mainly applicable to unsaturated soils with high degrees of saturation.

4. PARAMETER DETERMINATION

(1) Pore fluid volume compression modulus B_m

B_m is defined as

$$B_m = -\frac{du_m}{d\varepsilon_{v1}} = (1+e_0)\frac{du_m}{de_{a1}} \tag{18}$$

where e_0 is the initial void ratio and e_{a1} is the proportion of voids filled with air.

$$e_{a1} = (1-S_r)e \tag{19}$$

Here, S_r is the degree of saturation.

According to Boyle's Law,

$$e_{a1}(u_a + p_a) = e_{a0}(u_{a0} + p_a) \tag{20}$$

where p_a is the atmospheric pressure. Taking the derivative of each side of Equation (20), we obtain

$$\Delta e_{a1} = -\frac{(u_{a0}+p_a)e_{a0}}{(u_a+p_a)^2}\Delta u_a = -\frac{e_{a1}}{u_a+p_a}\Delta u_a = -\frac{e(1-S_r)}{u_a+p_a}\Delta u_a \tag{21}$$

Substituting (21) into (18), we obtain

$$B_m = (1+e_0)\frac{\Delta u_m}{\Delta e_{a1}} = \frac{(1+e_0)(u_a+p_a)}{e(1-S_r)}\frac{\Delta u_m}{\Delta u_a} \tag{22}$$

Based on the assumption that $u_w = u_a$,

$$B_m = \frac{(1+e_0)(u_m+p_a)}{e(1-S_r)} \tag{23}$$

(2) Variation in the degree of saturation, S_r

The variation in the degree of saturation over the time interval Δt can be expressed as

$$\Delta S_r = \frac{-S_r \Delta e_{a1}}{e} \tag{24}$$

Substituting Eq. (21) and assuming that $u_w = u_a$,

$$\Delta S_r = \frac{(1-S_r)S_r}{u_m+p_a}\Delta u_m \tag{25}$$

(3) Unit weight of mixed pore fluid r_m

$$r_m = S_r r_w \tag{26}$$

where r_w is the unit weight of pore water.

(4) Permeability coefficient of pore fluid K_m

Chang [21] presented an empirical equation of the following form that satisfactorily describes the relationships between the soil permeability, void ratio, and degree of saturation:

$$K = K_s G_e H_s \tag{27}$$

where K is the permeability of the unsaturated soil, K_s is the permeability of completely saturated soil, G_e is a factor whose value depends on the void ratio, and H_s is a factor whose value depends on the degree of saturation. According to Chang and Duncan's [14] investigation, G_e and H_s can be expressed by the following equations:

$$G_e = \frac{(1+e_0)e^3}{(1+e)e_0^3}\ ;\ \ H_s = S_r^3 \tag{28}$$

Therefore, the permeability coefficient of a mixed pore fluid K_m can be expressed as

$$K_m = K_s \frac{1+e_0}{1+e}\left(\frac{S_r e}{e_0}\right)^3 \tag{29}$$

5. VERIFICATION AND ANALYSIS

(1) Calculation example

The example of a vertical drain used by Teh [22] is considered here. The basic sizes are as follows: the diameter of the influence area of the drain D_e=3 m, the diameter of the smear zone d_s=0.2 m, and the diameter of the vertical drain d_w=0.1 m. The vertical drain system is converted into an equivalent rectangular parallelepiped influence area according to the theories of Miller [23] and Indraratna [24], as shown in Fig. 1. The corresponding sizes B_e (the width of the influence area of the drain), b_s (the width of the smear zone), and b_w (the width of the vertical drain) are shown in Fig. 1. The depth of the vertical drain H=5 m. Free displacement is assumed at the top boundary, and slip is assumed at the bottom and surrounding boundaries. The top, bottom, and surrounding boundaries are assumed to be impervious; only the mouth of the vertical drain is pervious. The soil properties are shown in Table 1.

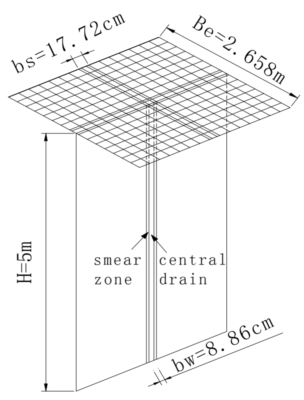

Fig. 1. Calculation example

Table 1. Soil properties used in the example

S_r (Initial degree of saturation)	85% or 95%
E (Modulus of elasticity)	10.0 MPa
ν (Poisson's ratio)	0.0
k_w (Permeability of the vertical drain)	10^{-4} m/s
k_s (Permeability of the smear zone)	2.5×10^{-9} m/s
k_h (Horizontal permeability of the undisturbed zone)	10^{-8} m/s
k_v (Vertical permeability of the undisturbed zone)	2.5×10^{-9} m/s
e_0 (Initial void ratio)	0.7
q_0 (Instantaneous load on top of the foundation)	100 kPa

(2) Analytical solution for saturated soil

To verify the reliability of the numerical analysis theory, the numerical solution was compared to the analytical solution for saturated soil. According to Hansbo's [25] analytical theory of vertical drains, the average degree of consolidation at a depth of z can be expressed as

$$\overline{U}_h = 1 - \exp\left(-\frac{8c_h t}{\mu D_e^2}\right) \tag{30}$$

In which $c_h = \dfrac{k_h}{m_v r_w}$; $m_v = \dfrac{(1+\nu)(1-2\nu)}{E(1-\nu)}$; $\mu = \mu_1 + \mu_2 + \mu_3$;

$$\mu_1 = \frac{D_e^2}{D_e^2 - d_w^2}\left[\ln\left(\frac{D_e}{d_s}\right) + \frac{k_h}{k_s}\ln\left(\frac{d_s}{d_w}\right) - \frac{3}{4}\right];\quad \mu_2 = \frac{\pi k_h}{q_w} z(2H - z)\left(1 - \frac{d_w^2}{D_e^2}\right);$$

$$\mu_3 = \frac{d_s^2}{D_e^2 - d_w^2}\left(1 - \frac{d_s^2}{4D_e^2}\right) + \frac{k_h}{k_s}\frac{d_w^2}{D_e^2 - d_w^2}\left(\frac{d_s^4 - d_w^4}{4D_e^2 d_w^2} - \frac{d_s^2}{d_w^2} + 1\right);\quad q_w = k_w \frac{\pi d_w^2}{4}$$

(3) Calculation and analysis

Two cases were considered. The degree of saturation of the soil is 85% in the first case and 95% in the second case. The average degree of consolidation at a depth of zero is plotted for the first case in Fig. 2. The plot indicates the following: (1) the numerical analysis theory is applicable and reliable; (2) the initial deformation of unsaturated soils is greater than that of saturated soils, possibly due to the compaction of pore air under instantaneous loads; and (3) the average rate of consolidation of unsaturated soils is slower than that of saturated soils (the saturated soil requires approximately 10 days to reach a U of 80%, whereas the unsaturated soil requires approximately 30 days).

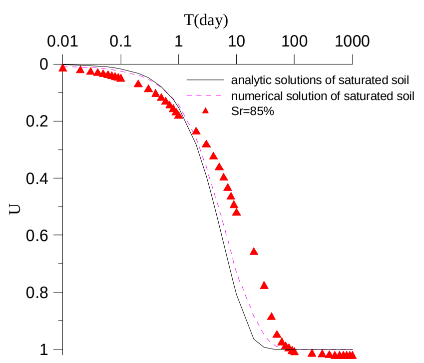

Fig. 2. Degree of consolidation vs. time

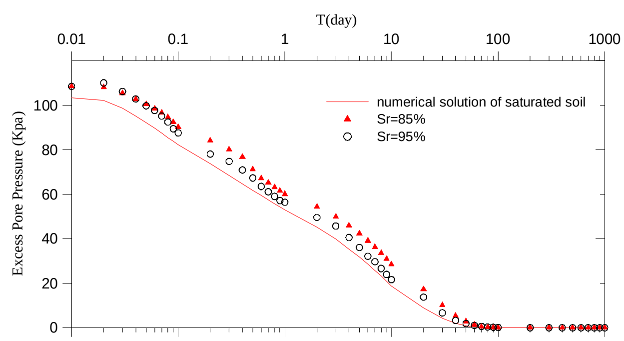

Fig. 3. Pore fluid pressure vs. time curve for point A

The pore fluid pressures at point A (see Fig. 1) are plotted for both cases in Fig. 3. The rate of dissipation of the pore fluid pressure for the unsaturated soil is considerably slower than that for saturated soil.

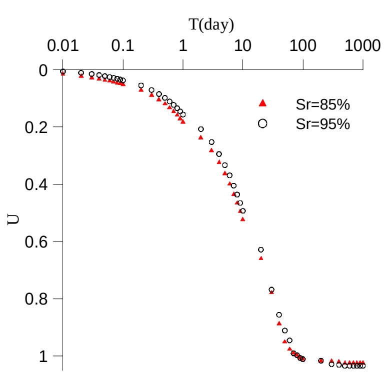

Fig. 4. Degree of consolidation vs. time

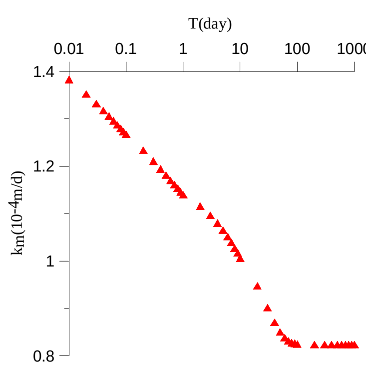

Fig. 5. Permeability coefficient vs. time

The effect of the degree of saturation on the pore fluid pressure and displacement were considered (see Figs. 3 and 4). The average degree of consolidation of the unsaturated soil decreases with an increasing degree of saturation, as shown in Fig. 4. The rate of dissipation of the pore fluid pressure for unsaturated soil slows with a decreasing degree of saturation, as shown in Fig. 3. The variation in the permeability of the pore fluid was also considered, as shown in Fig. 5. The vertical permeability coefficient K_m of the undisturbed zone decreases considerably.

6. CONCLUSION

Biot's theory for three-dimensional consolidation was extended to unsaturated soils by treating the pore water and pore air as a mixed pore fluid. The theory was then formulated for finite element analysis. The finite element formulation considers variations in the permeability and compressibility of the mixed pore fluid with changes in the void ratio and degree of saturation. An example of settlement of a vertical drain was investigated, and the results indicate that the numerical analysis theory is applicable and reliable. This theory is most appropriate for applications to unsaturated soils with high degrees of saturation.

The results demonstrate that the rate of consolidation of unsaturated soils is clearly slower than that of saturated soils, the rate of dissipation of the pore fluid pressure for unsaturated soil is considerably slower than that of saturated soil, and the permeability of the pore fluid decreases during consolidation. These phenomena may have adverse effects on the safety and reliability of engineering designs involving soil consolidation calculations. Therefore, the use of the theory of three-dimensional consolidation of unsaturated soils is suggested to obtain more reliable predictions of unsaturated soil consolidation.

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