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"Research Note"

CANONICAL FORMS FOR CALCULATING THE EIGENFREQUENCIES AND BUCKLING LOADS OF SYMMETRIC SPACE TRUSSES*

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Abstract– For dynamic analysis of structure, the calculation of eigenvalues and eigenvectors is necessary. When the structural models are symmetric, such calculation can be simplified using some of the concepts of graph theory. In this paper, two methods are presented for eigensolution of space Truss. The first method uses a graph model and employs a decomposition and healing process for factorization of the graph model and calculating the eigenvalues of graph. The second approach uses the canonical forms for the construction of submatrices, from which, the eigenvalues can be obtained. Both methods lead to identical results.

Keywords– Free vibration, Forced vibration, canonical forms, graph theory, buckling load, space trusses

1. INTRODUCTION

Symmetry has been widely used in science and engineering [1-5]. A thorough review can be found in the work of Kangwai et al. (1999). Methods are developed for decomposing and healing the graph models of structures, in order to calculate the eigenvalues of matrices and graph matrices with special patterns [6-9]. For symmetric structures, it is advantageous to use this property for studying the free vibration and stability analysis of these structures. In this paper, special canonical forms, namely the Form A and the Form B matrices previously developed for planar trusses are extended for the free vibration analysis of three dimensional trusses. These forms are used for efficient calculation of the eigenvalues of symmetric structures. Here, two methods are presented, the first approach uses graph theory and the second method has an algebraic nature and the degrees of freedoms are ordered, employing a combination of symmetric and anti-symmetric to form the canonical forms. In the first method, special definitions are used to form the Laplacian matrices of the required canonical forms, and then using special operators the submatrices are constructed from which the eigenvalues are easily calculated. In the second approach, the stiffness matrix is formed for the entire structure and using the symmetry and anti-symmetry for its entries the required canonical forms are constructed, and employing special operators the submatrices are formed from which the eigenvalues of the entire structure are calculated. Finally, examples are studied to illustrate a step by step process of the presented methods.

2. DETERMINATION OF SYMMETRIC FORMS FOR A AND B BY USING THE GRAPHS THEORY RELATIONS

a) Determining the form of a using graphs theory relations

In a symmetric graph if the left sides of symmetry axis nodes are numbered at first and then the right side of symmetry axis nodes, the following steps are applied:

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1. If there is a symmetric graph, select two symmetric node categories (N_1, N_2).Noting that both of these categories are symmetric, each of them can be divided into two sub collections.

$$N_{1R}, N_{1L}, N_{2R}, N_{2l} \tag{1}$$

2. D matrix for the whole graph can be defined as follows:
3. Two sub graphs are formed as follows:

3.1. Sub graph G_1 : deletes all members that connect N_{1R} to N_2 .

3.2. Sub graph G_2 : this is a graph that includes connecting members between nodes of N_{1R} and N_2

3.3. G graph which is a combination of the two upper graphs is formed, but G_2 members are represented by dashed line.

$$L' = D' - A' \qquad D' = [d_{ij}] \quad \begin{cases} i = j \rightarrow d_{ii} = 1 - i \\ i \neq j \rightarrow d_{ij} = 0 \end{cases} \tag{2}$$

Dashed line loops are recognized by negative sign, and i is the number of node loops.

$A' = \lfloor a_{ij} \rfloor$ is number of members whose direction is from i to j (members with no sign means members with two directions).

When members are represented by dashed line it means they have a negative sign. As a result, modified symmetric Laplacian matrix is obtained in the following form:

$$\mathrm{L}'_{\mathrm{N\times N}} = \begin{bmatrix} [\mathrm{A}]_{\mathrm{n\times n}} & [\mathrm{C}]_{\mathrm{n\times n}} & [\mathrm{D}]_{\mathrm{n\times n}} & [\mathrm{F}]_{\mathrm{n\times n}} \\ [\mathrm{C}]_{\mathrm{n\times n}} & [\mathrm{B}]_{\mathrm{n\times n}} & [\mathrm{F}]_{\mathrm{n\times n}} & [\mathrm{E}]_{\mathrm{n\times n}} \\ [\mathrm{D}]_{\mathrm{n\times n}} & -[\mathrm{F}]_{\mathrm{n\times n}} & [\mathrm{A}]_{\mathrm{n\times n}} & -[\mathrm{C}]_{\mathrm{n\times n}} \\ -[\mathrm{F}]_{\mathrm{n\times n}} & [\mathrm{E}]_{\mathrm{n\times n}} & -[\mathrm{C}]_{\mathrm{n\times n}} & [\mathrm{B}]_{\mathrm{n\times n}} \end{bmatrix}_{\mathrm{N\times N}} \tag{3}$$

Matrices are symmetric, except sub matrix F that is anti symmetric. By applying row and column operators, matrix L' can be changed to an upper rectangular matrix [6-7].

$$[L'] = \left[\begin{array}{cc:cc} A+D & C-F & D & F \\ C+F & B-E & F & E \\ \hdashline 0 & 0 & A-D & -C-F \\ 0 & 0 & F-C & B+E \end{array}\right] \tag{4}$$

$$Det[L'] = \underbrace{Det\begin{bmatrix} A+D & C-F \\ C+F & B-E \end{bmatrix}}_{S} \times \underbrace{Det\begin{bmatrix} A-D & -C-F \\ -C+F & B+E \end{bmatrix}}_{T} \tag{5}$$

$$\lambda(L') = \lambda(S) \cup \lambda(T) \tag{6}$$

b) Determination of B form by using graphs theory relations

At first, the left side of symmetry axis nodes are numbered, then the right side symmetry axis nodes, and finally nodes that are mounted on the axis symmetry.

1. If there is a symmetric graph two categories of symmetric nodes (N_1, N_2) are selected. Regarding the symmetry of these categories, each of them can be divided into two collections.

$$N_{1R}, N_{1L}, N_{1C}, N_{2R}, N_{2l}, N_{2C} \tag{7}$$

N_{1C}, N_{2C} are nodes of two collections (N_1, N_2) which are on the symmetric axis.

2. Matrix D can be defined for the whole graph as follows:
3. Two sub graphs are formed as follows:

3.1. Sub graph G_1 : members that connect N_{1R} to N_2 and N_{1C} to N_{2R} are deleted.

3.3. Sub graph G_2 : members that connect N_{1R} to N_2 and N_{1C} to N_{2R} form a graph.

3.3. Graph G which is a combination of the two upper graphs is formed but the members of sub graph G_2 are represented by dashed line.

$$L' = D' - A' \qquad D' = [d_{ij}] \quad \begin{cases} i = j \rightarrow d_{ii} = 1 - i \\ i \neq j \rightarrow d_{ij} = 0 \end{cases} \tag{8}$$

Dashed line loops are assumed with a negative sign, and i is the number of node loops.

$A' = \lfloor a_{ij} \rfloor$ are members whose directions are from i to j (no direction members means double sided members).

Members that are represented by dashed line have negative signs.

D' is modified degree matrix, A' is modified adjacency matrix and L' is modified Laplacian matrix.

Notice: between N_{1C} and N_{2C} there shouldn't be any connecting member.

Example 1: A symmetric graph as shown in Fig. 1 is considered. Its eigenvalues are calculated as follows:

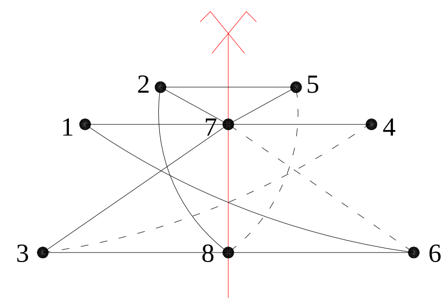

Fig. 1. Symmetry graph

$$L' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Sub graphs:

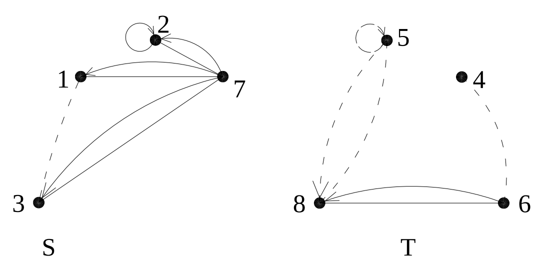

Fig. 2. Formation of S, T sub graphs

$$L'_S = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 1 \end{bmatrix} \qquad L'_T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\lambda_S = [3.7785, -1.4898, 0, 0.7108] \qquad \lambda_T = [-1, 0.5858, 2, 3.4142]$$

$$\lambda_{L'} = \lambda_S \cup \lambda_T = [3.7785, -1.4898, 0, 0.7108, -1, 0.5858, 2, 3.4142]$$

3. DETERMINATION OF SYMMETRIC FORMS OF A AND B BY USING DIRECT AND REVERSE SYMMETRIC RELATIONS

a) Determination of symmetric form of A by using direct and reverse symmetric relations

Assume coordinate system in the three dimensional space is shown with numbers 1-2-3. If there is a symmetry plane in this space and this plane is parallel to 3-2 surface, then mirror of every vector in 3-2 plane regarding the symmetry plane will be itself, while the mirror of every vector parallel to vector 1 regarding the symmetry plane will be in inverse direction of 1 (Fig. 3).

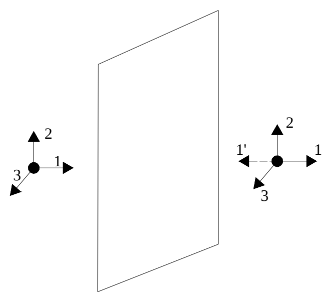

Fig. 3. Symmetry plane

Now, without attention to symmetry, each distinctive matrix (stiffness, mass, damping) for a structure with a part of it in one side of an assumed plane parallel to 2-3 surface, another part in the other side of the this surface, and DOFs for each node of it defined parallel to 1-2-3 axis, can be written in the following general form:

Table 1. The (stiffness, mass, damping …) matrix of structure

		Degree of freedom of the left side			Degree of freedom of the right side		
		1Direction	2Direction	3Direction	1Direction	2Direction	3Direction
Degree of freedom of the left side	1Direction	A	D	F	G	J	L
	2Direction	D^T	B	E	M	H	K
	3Direction	F^T	E^T	C	N	O	I
Degree of freedom of the right side	1Direction	G^T	M^T	N^T	P	S	U
	2Direction	J^T	H^T	O^T	S^T	Q	T
	3Direction	L^T	K^T	I^T	U^T	T^T	R

There are no nodes on the former assumed plane. In this case matrices R, Q, P, C, B, A are symmetric. Now, if this plane is a symmetric plane, first, it should be noted that number of DOFs in the 1, 2, 3 directions in the right side of this plane is equal to the same in the left side of this plane. Therefore matrices I, H, G will be square.

It should be noted that the obtained relations for this load are more general than the obtained relations for two-dimensional case. One of the differences between these two cases is that, while in two-dimensional cases it is considered that the number of degrees of freedom is equal in 1, 2, and 3 directions in three-dimensional cases, this consideration is not valid and the problem will be investigated in a more general situation. It is assumed that the DOF in direction 1 in each side of symmetry plane is equal to m, the DOF in direction 2 in each side of symmetry plane is equal to n, and the DOF in direction 3 in each side of symmetry plane is equal to o. This means some DOFs can be active or inactive, while the symmetry conditions in both sides of symmetry plane should be retained. Ordering in both sides of symmetry plane is as follows: First, all DOFs in direction1 in left side of symmetry plane are numbered, then the numbering of the DOFs in direction 2 in left side of symmetry plane, and finally DOFs in direction 3 in left side of symmetry plane are numbered. A similar numbering is then performed for the DOFs of the right side.

Example 2: Consider indeterminate 3D truss as shown in Fig. 4 which has 20 elements; frequencies and buckling loads of the structure are calculated:

$$E = 2.07 \times 10^7\ kN/m^2 \quad I = 100\,cm^2 \quad \rho = 7800\,kg/m^3 \quad A = 10\,cm^2$$

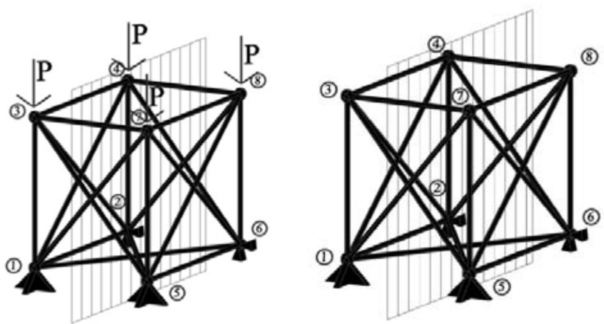

Fig. 4. Symmetry and indeterminate 3D truss

All DOFs of nodes of 1, 5 and 1, 2 DOFs of nodes 2, 6 are inactive. The only symmetry plane is 2-3 plane. At first, the left side symmetric plane nodes are numbered then the right side of symmetric plane nodes. The numbering of the right side is the same as the left side. In the above truss the number of DOF in the directions of 1, 2 (m,n) in each side of symmetry axis is equal to 2 and the number of DOFs in the direction 3 (o) in each side of symmetric axis is equal to 3.

$$\omega_D = [32.35, 28.618, 24.076, 21.771, 10.606, 8.197, 25.51]\ rad/\sec$$

$$\omega_C = [35.172, 8.944, 31.516, 28.663, 26.432, 20.964, 22.461]\ rad/\sec$$

$$\omega = \omega_D \cup \omega_C = \begin{bmatrix} 32.35, 28.618, 24.076, 21.771, 10.606, 8.197, 25.51 \\ 35.172, 8.944, 31.516, 28.663, 26.432, 20.964, 22.461 \end{bmatrix}\ rad/\sec$$

$$P_{cr(s)} = 51900\ kN \qquad P_{cr(T)} = 72000\ kN \qquad P_{cr} = 51900\ kN$$

b) Determination of symmetric B form by using direct and reverse symmetric relations

Now a case is considered with nodes on symmetry plane (Fig. 5):

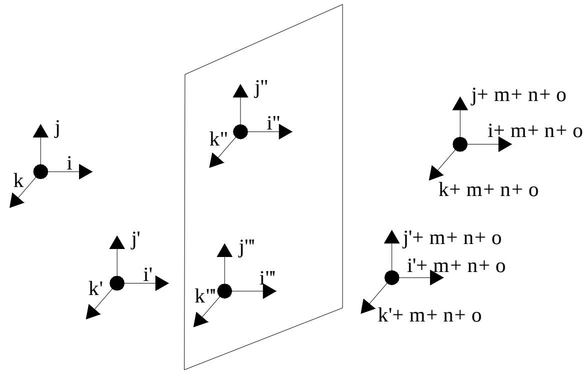

Fig. 5. Symmetry plane with nodes on itself

It is considered that the number of DOFs in direction 1 for each side of symmetry plane is equal to m, number of DOFs in direction 2 for each side of symmetry plane is equal to n, and number of DOFs in direction 3 for each side of symmetry plane is equal to o. The DOFs in direction 1 on the symmetry plane is m', the DOFs in direction 2 on the symmetry plane is n', the DOFs in direction 3 on the symmetry plane is o'. Ordering in two sides of symmetry plane is as follows: First, all DOFs in direction 1 in the left side of symmetry plane are numbered, DOFs in direction 2 in left side of symmetry plane are numbered, and then DOFs in direction 3 in left side of symmetry plane are numbered. After that, a similar procedure for numbering is performed for DOFs of the right side.

Example 3: Consider indeterminate 3D truss as shown in the Fig. 6, frequencies and buckling load of the structure are calculated as follows: $E = 2.07 \times 10^7\ kN/m^2$ $I = 100\ cm^4$ $\rho = 7800\ kg/m^3$ $A = 10\ cm^2$

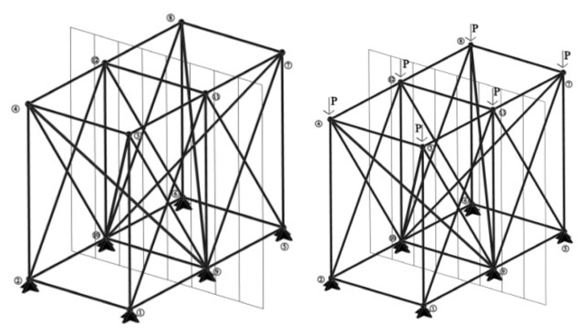

Fig. 6. Symmetry and indeterminate 3D truss

$$\omega_S = \{104.38, 87.42, 84.48, 65.06, 59.26, 24.06, 22.38, 10.77\}\ rad/\sec$$

$$\omega_T = \begin{Bmatrix} 86.48, 74.31, 48.11, 67.43, 67.48, 61.48, 56.22, 55.55 \\ 12.53, 11.13 \end{Bmatrix} rad/\sec$$

$$\omega = \omega_S \cup \omega_T = \begin{Bmatrix} 104.38, 87.42, 84.48, 65.06, 59.26, 24.06, 22.38, 10.77 \\ 86.48, 74.31, 48.11, 67.43, 67.48, 61.48, 56.22, 55.55 \\ 12.53, 11.13 \end{Bmatrix} rad/\sec$$

$$p_{cr(S)} = 17800\ kN \qquad P_{cr(T)} = 16000\ kN \qquad P_{cr} = 16000\ kN$$

4. CONCLUSION

Large eigenvalue problems arise in many scientific and engineering applications. In such problems, the numerical determinants to be calculated are very complicated, and special approaches for these problems are more effective. Eigenvalue problems include combinational and mechanical problems. Here only mechanical problems are investigated. This is performed by using special symmetry forms, and large problems are transformed into smaller sub problems. This reduction in size leads to higher accuracy and reduction of the calculation time for the solution of these problems.

The methods presented for decomposition and healing for 3D trusses reduce the computational storage and time for eigen solution of structures. From the two methods, if the decomposed substructures are themselves symmetric, then further decomposition is possible.

Although this study is made for the free vibration and the forced vibration of the trusses, the saving in computer time increases by the size of the structural models.

REFERENCES

1. Hargittai, I. (1986). *Symmetry; unifying human understanding*. Pergamon Press Ltd, UK.
2. Gruber, B. (1995). *Symmetries in Science VIII*. Plenum Press, NY.
3. Zingoni, A. (1995). *Group-theoretical applications in solid and structural mechanics: a review, Chapter 12 in Computational Structures Technology*. Edited by BHV Topping and Z. Bittnar, Saxe-Coburg Publication, UK.
4. Kaveh, A. & Laknejadi, K. (2011). A hybrid multi-objective optimization and decision making procedure for optimal design of truss structures. *Iranian Journal of Science & Technology, Transactions of Civil Engineering*, Vol. 35, No. C2, pp. 137-154.
5. Kaveh, A. & Shakouri Mahmud Abadi, A. (2010). Harmony search algorithm for optimum design of slab formwork. *Iranian Journal of Science & Technology, Transaction B: Engineering*, Vol. 34, No. B4, pp. 335-351.
6. Kangwai, R. D., Guest, S. D. & Pellegrino, S. (1999). An introduction to the analysis of symmetric structures. *Computers and Structures*, Vol. 71, pp. 671-688.
7. Kaveh, A. & Shahryari, L. (2010). Eigenfrequencies of symmetric planar trusses via weighted graph symmetry. *Engineering Computations*, Vol. 27, pp. 409-439.
8. Kaveh, A. & Sayarinejad, M. A. (2004). Eigensolutions for factorable matrices of special patterns. *Communications in Numerical Methods in Engineering*, Vol. 20, pp. 133-146.
9. Kaveh, A. & Shahryari, L. (2007). Buckling load of planar frames with semi-rigid joints using weighted symmetric graphs. *Computers and Structures*, Vol. 85, pp. 1704-1728.