OPTIMAL DESIGN OF CANTILEVER RETAINING WALLS
USING RAY OPTIMIZATION METHOD*

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Abstract— Earth retaining structures are referred to those structures which can control backfill heights that are just about to slide. Some examples of these structures are gravity and cantilever retaining walls. The cantilever retaining walls were utilized after the introduction of the reinforced-concrete construction technique. In the previous studies, the optimization of the retaining walls has been accomplished by quasi-static methods; however, in this paper a pseudo-dynamic approach is utilized. The advantage of the pseudo-dynamic analysis is that the phase difference effects and time can be entered in the design of retaining walls as the dynamic characteristics of the earthquake loading. Here, by optimizing a cantilever retaining wall via a recently developed method, so-called Ray Optimization, the design controlling parameters are investigated. Ray Optimization method is a multi-agent optimization method which is inspired from the concept of light refraction. In this method by moving the agents to new positions, the optimal solution is found.

Keywords— Pseudo-dynamic approach, cantilever retaining wall, ray optimization

1. INTRODUCTION

Earth retaining structures are referred to those structures which can control backfill heights that are just about to slide. These structures are used when the gravity retaining walls are uneconomical. Though the retaining walls are often out of sight of the public, like other tall building structures and bridges, they have an important role in the societies.

If there is a mistake in the design of retaining walls, it causes a great deal of catastrophic damages. One can perform the analysis and design retaining walls by static, quasi-static, pseudo-dynamic and dynamic approaches. In order to design this structure by static approach, the Rankine or Coulomb theory can be utilized [1]. In this manner, the backfill thrust can be related to some coefficients, and ultimately using an equation, this pressure can be calculated. Another method that can be used is the quasi-static methods. In the quasi-static approach, the transient earthquake force and static thrust are simultaneously imposed on the retaining wall as an equivalent static force.

This method is based on the plasticity theory and is essentially an extension of the Coulomb sliding wedge theory. The pioneers of this method are Mononobe and Matsuo [2] and Okabe [3] and their work is known as Mononobe-Okabe method. In the quasi-static methods, the dynamic natural of the earthquake loading is considered to some extent. Now if one can consider some dynamic properties like phase difference effect and time in the backfill of the retaining walls, he or she will achieve a pseudo-dynamic approach. Methods like Steedman and Zeng [4] and Choudhury and Nimbalkar [5] are examples of this approach. The last method for the analysis and design of the retaining walls is dynamic one. The

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performed analyses in this method are generally based on finite element method in the semi-infinite environment, thus because of high computational time, dynamic method is generally utilized for scientific purposes [6].

To obtain a design with minimum cost and time consumption, optimization methods must be used. For examples Kaveh and Behnam [7] used Charged System Search algorithm for optimization of gravity retaining walls, and Kaveh and Shakouri [8] employed harmony search algorithm for optimization of cantilever retaining walls. Both of these works use quasi-static approach for determining the active earth pressure behind the retaining walls. However, in this paper the pseudo-dynamic approach is utilized for determining this force. This optimization is performed by a new meta-heuristic method, so-called Ray optimization (RO), Ref. [9].

In this article after introducing the pseudo-dynamic approach which is offered by Choudhury and Nimbalkar [5], the concepts of this new optimization method are presented. The optimization basics of a cantilever retaining wall are gathered in section 4. In section 5, the optimal design of a cantilever retaining wall under different earthquake loading conditions is performed. The paper is concluded in section 6.

2. PSEUDO-DYNAMIC METHOD OF SEISMIC ACTIVE EARTH PRESSURE BEHIND RETAINING WALLS

A common method for determining the distribution of seismic earth pressure is Mononobe-Okabe method [10]. This approximate method offers a linear pressure distribution behind the retaining wall and does not consider the time as a natural feature of the earthquake loading. Pseudo-dynamic approach provides a condition in which the realistic non-linear distribution of active earth pressure can be presented by considering the finite shear wave propagation. Because of the existence of the finite shear wave propagation in the problem, time and phase difference, which are two important characteristics of the earthquake loading, can have their roles in the calculation of the seismic active earth pressure. Choudhury and Nimbalkar [5] performed such analysis in which a wide range of parameters were considered. These were wall friction angle ($\delta$), soil friction angle ($\phi$), shear wave velocity ($V_s$), primary wave velocity ($V_p$) and horizontal and vertical ground accelerations $a_h$ and $a_v$, respectively.

The following definitions and concepts are taken mainly from Ref. [5].

Consider the fixed base vertical rigid retaining wall AB of height H as shown in Fig. 1. The wall supports a cohesionless backfill material with horizontal ground. In the present study, two types of wave velocities are considered:

1. The shear wave velocity $V_s = \sqrt{G/\rho}$, with $\rho$ and $G$ being the density and shear modulus of the backfill material.

2. The primary wave velocity $V_p = \sqrt{G(2-2\nu)/\rho(1-2\nu)}$, with $\nu$ being the Poisson’s ratio of the backfill.

![Fig. 1. Model of the retaining wall considered for computation of pseudo dynamic active earth pressure [5]](image)
The shear waves are assumed to act within the soil media due to earthquake loading. For most geological materials $V_p / V_s = 1.87$, Ref. [11]. The period of lateral shaking is considered in the analysis as $T = \frac{2\pi}{\omega} = \frac{4H}{V_s}$, where $\omega$ is the angular frequency [10]. A planer rupture surface inclined at an angle $\alpha$ with the horizontal is considered in the analysis.

If the base of the wall is subjected to harmonic horizontal seismic acceleration of amplitude $a_h g$, where $g$ is the acceleration due to gravity and harmonic vertical seismic acceleration of amplitude $a_v g$, the acceleration at any depth $z$ and time $t$, below the top of the wall can be expressed as:

$$a_h [z, t] = a_h \sin [\omega (t - \frac{H - z}{V_s})]$$

$$a_v [z, t] = a_v \sin [\omega (t - \frac{H - z}{V_p})]$$

After some calculations, Choudhury and Nimbalkar define the seismic active earth pressure coefficient $K_{ae}$ as:

$$K_{ae} = \frac{\sin (\alpha - \phi)}{\tan \alpha \cos (\delta + \phi - \alpha)} + \frac{k_h}{2\pi^2 \tan \alpha} \left( \frac{TV_p}{H} \right) \times \frac{\cos (\alpha - \phi)}{\cos (\delta + \phi - \alpha)} \times m_1$$

$$- \frac{k_v}{2\pi^2 \tan \alpha} \left( \frac{TV_p}{H} \right) \times \frac{\sin (\alpha - \phi)}{\cos (\delta + \phi - \alpha)} \times m_2$$

$$m_1 = \left[ 2\pi \cos 2\pi \left( \frac{t}{T} - \frac{H}{TV_p} \right) + \left( \frac{TV_s}{H} \right) \sin 2\pi \left( \frac{t}{T} - \frac{H}{TV_v} \right) - \sin 2\pi \left( \frac{t}{T} \right) \right]$$

$$m_2 = \left[ 2\pi \cos 2\pi \left( \frac{t}{T} - \frac{H}{TV_p} \right) + \left( \frac{TV_p}{H} \right) \sin 2\pi \left( \frac{t}{T} - \frac{H}{TV_v} \right) - \sin 2\pi \left( \frac{t}{T} \right) \right]$$

where $k_h$ and $k_v$ are $a_h / g$ and $a_v / g$, respectively. Finally the seismic active earth pressure $P_{ae}$ can be calculated by:

$$P_{ae} = \frac{1}{2} K_{ae} \gamma H^2$$

where $\gamma$ is the unit weight of the backfill. For obtaining the maximum value of $K_{ae}$, it is necessary to maximize Eq. (2) with respect to $t/T$ and $\alpha$ (see Appendix A).

By differentiating Eq. (3), the seismic active earth pressure distribution is obtained as:

$$p_{ae}(t) = \frac{\gamma z}{\tan(\alpha)} \frac{\sin (\alpha - \phi)}{\cos (\delta + \phi - \alpha)} + \frac{k_h \gamma z}{\tan(\alpha) \cos (\delta + \phi - \alpha)} \sin [\omega (t - \frac{z}{V_s})]$$

$$- \frac{k_v \gamma z}{\tan(\alpha) \cos (\delta + \phi - \alpha)} \sin [\omega (t - \frac{z}{V_p})]$$

### 3. RAY OPTIMIZATION

Consider a light ray which is crossing the transparent medium $K$ with the vector $V_i^k$, Fig. 2. When this ray reaches to the point $X_i^k$, after refracting it enters to the darker medium $K+1$ and continues its path with the vector $V_i^{k+1}$. The direction of $V_i^{k+1}$ is dependent on the direction of $n$ and the refraction index ratio $(n_g / n_i)$. For determining the direction of this vector, refer to Appendix B.
Ray Optimization method is a multi-agent optimization method which is inspired from the concept of light refraction [9]. In this method by moving the agents to new positions, the optimal solution is found. Thus, if the movement vector for the \(i\)th agent in the \(k\)th iteration is \(V_i^k\) and the current position of this agent is \(X_i^k\), it can be moved to its new position by \(V_i^{k+1}\). The refraction index ratio for this method is selected as 0.45. The direction of \(n\) passes through two points. The beginning point is \(O_i^k\) and final point is \(X_i^k\). \(O_i^k\) is defined as:

\[
O_i^k = \frac{(ite+k).GB + (ite-k).LB_i}{2.ite}, \tag{5}
\]

where \(GB_i\) and \(LB_i\) are the so-far best position and goal function value obtained by all of the agents and \(i\)th agent, respectively.

If the number of variables is greater than 3, for using the ray tracing concept the search space can be divided to a number of 2D and or 3D spaces. In general, if \(N\) is an even number, the search space is divided to \((N/2)\) 2D spaces and if \(N\) is an odd number, the search space is divided to \((N-3)/2\) 2D space(s) and a one 3D space. Each of these 2D or 3D spaces is named sub-space. With this description, \(V_{l,i}^k\) is the movement vector of the \(l\)th sub-space which belongs to the \(i\)th agent in the \(k\)th iteration and \(v_{l,i,j}^\theta\) is the \(j\)th component of the movement vector of the \(l\)th sub-space which belongs to \(i\)th agent in the \(k\)th iteration.

The steps of Ray Optimization algorithm are as follows:

**a) Scattering and evaluation step**

Based on Eq. (6), scatter the agents in the search space, randomly.

\[
x_{i,j}^\theta = x_{j,min} + rand \times (x_{j,max} - x_{j,min}), \tag{6}
\]

where, \(x_{i,j}^\theta\) is the \(j\)th component of the \(i\)th agent. \(x_{j,min}\) and \(x_{j,max}\) are the allowable minimum and maximum values of the \(j\)th component. Here, \(rand\) is a random number distributed 0 through 1. After scattering, evaluate the value of goal function for each agent. Then, save the position and goal function value of each agent and the best position and goal function value of the best agent as \(LB_i\) and \(GB_i\), respectively.

Make a movement vector for each agent based on Eq. (7).

\[
v_{i,j}^\theta = -1 + 2 \times rand, \tag{7}
\]

where, \(v_{i,j}^\theta\) is the \(j\)th component of the \(i\)th agent. Finally, based on the sub-space grouping, convert 2D and 3D movement vector to normalized ones.

**b) Movement vector and motion refinement step**

Move the agents to their new positions based on their movement vectors. If an agent violates the allowable boundaries, modify the length of its movement vector. The new length of movement vector is...
equal to 0.9 times the distance of current agent position and the intersection with boundary. After modifying the movement vector, evaluate the goal function of each agent and update $GB$ and $LB_i$.

c) Cockshy point making and convergent step

Determine $O_i^k$ for each agent. Then, based on Eq. (8), obtain the new movement vector. In this equation $stoch$ and $d$ are 0.35 and 7.5, respectively.

$$V_{i,j}^{k+1} = \frac{V_{i,j}^{k+1}}{\text{norm}(V_{i,j}^{k+1})} \times \frac{a}{d} \times \text{rand}$$

each component of $V_{i,j}^{k+1}$ is calculated as below:

$$V_{i,j}^{k+1} = -1 + 2 \times \text{rand}$$

$$a = \left(\sum_{j=1}^{g} (x_{j,max} - x_{j,min})^2\right)^{\frac{1}{2}}$$

$$n = \begin{cases} 2 & \text{for two variable groups} \\ 3 & \text{for three variable groups} \end{cases}$$

$$X_{i,j}^k = O_{i,j}^k \rightarrow V_{i,j}^{k+1} = \frac{V_{i,j}^k}{\text{norm}(V_{i,j}^k)} \times \text{rand} \times 0.001$$

d) Finish or redoing step

If the finishing criterion of algorithm is fulfilled the search procedure terminates, otherwise the algorithm returns to the second step and continues the search. The finishing criterion can be a specific number of iterations for obtaining the optimal solution.

4. THE BASICS OF THE OPTIMIZATION OF CANTILEVER RETAINING WALL.

In the prior sections, pseudo-dynamic analysis of Choudhury and Nimblekar and Ray Optimization method were introduced. In this section, the basics of the optimal design of cantilever retaining wall are introduced.

In this problem, similar to Ref. [8], the cost of consumed concrete and steel is considered as goal function.

$$Q = V_{\text{conc}} \times (C_1 + C_2) + W_{\text{steel}} \times (C_3 + C_4)$$

By considering $\bar{Q} = Q / (C_1 + C_2)$, the goal function is converted to:

$$\bar{Q} = V_{\text{conc}} + W_{\text{steel}} \times \frac{C_3 + C_4}{C_1 + C_2}$$

Where $V_{\text{conc}}$ and $W_{\text{steel}}$ are the volume of concrete and the weight of reinforcement steel in the unit length (m$^3$/m and kg/m), $C_1$ and $C_2$ are the cost of the concrete and steel ($$/m^3$$ and $$$/kg), $C_3$ and $C_4$ are the cost of concreting and erecting reinforcement ($$/m^3$$ and $$$/kg). Experiences show the value of $C_1 + C_2$ is in the range of 0.035 to 0.045 and in this paper 0.04 is selected. The design variables in this problem are the
thickness of top stem ($T_1$), the thickness of the key and bottom stem ($T_2$), the toe width ($T_3$), the heel width ($T_4$), the height of top stem ($T_5$), the footing thickness ($T_6$) and the key depth ($T_7$), Fig. 3.

In this problem, there are two groups of constraints. The first group is about the stability of the cantilever retaining wall under exerted forces. This group is gathered from Ref. [12]. The second one is about the shear and flexural strength which is gathered from Ref. [13]. For more details of constraints see Appendix C.

For the sake of simplicity, the penalty approach is used for constraint handling. In using the penalty function, if the constraints are not violated, the penalty will be zero; otherwise the value of the penalty is calculated by dividing the violation of the allowable limit to the limit itself.

In the process of the optimization, the required rebar based on the ultimate moment, $M_u$, in each critical section is calculated and then the total weight of rebar for the cantilever retaining wall is obtained.

5. A NUMERICAL EXAMPLE

In this section, the optimum design of a cantilever retaining wall under 7 earthquake dynamic loading conditions are provided. $K_{ae}$ and the related parameters are taken from Appendix A. Based on the suggestion of Ref. [5]; $\frac{H}{TV}$ and $\frac{H}{TV^e}$ are selected as 0.25 and 0.1337, respectively. The allowable ranges of the design variables are given in Table 1. It should be noticed that for obtaining an optimum design in the case of $k_h=0.2$ and $k_v=0.2$, the maximum allowable value of the second variable is increased from 0.6 to 1.1m. The stem length of the cantilever retaining wall is constant and is equal to 6.1m. The other properties of this problem are gathered in Table 2.
Table 1. Upper and lower bounds for design variables

<table>
<thead>
<tr>
<th>Design variable</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$T_4$</th>
<th>$T_5$</th>
<th>$T_6$</th>
<th>$T_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper bound</td>
<td>0.3m</td>
<td>0.3m</td>
<td>0.45m</td>
<td>1.8m</td>
<td>1.5m</td>
<td>0.3m</td>
<td>0.2m</td>
</tr>
<tr>
<td>Lower bound</td>
<td>0.6m</td>
<td>0.6m</td>
<td>1.2m</td>
<td>3m</td>
<td>6.1m</td>
<td>0.9m</td>
<td>0.9m</td>
</tr>
</tbody>
</table>

Table 2. Properties of the numerical example

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_b$</td>
<td>20 kN/m$^3$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>30 (°)</td>
</tr>
<tr>
<td>$q_u$</td>
<td>300 kN/m$^2$</td>
</tr>
<tr>
<td>$f_c'$</td>
<td>25 MPa</td>
</tr>
<tr>
<td>$\gamma_c$</td>
<td>24 kN/m$^3$</td>
</tr>
<tr>
<td>$f_y$</td>
<td>420 MPa</td>
</tr>
<tr>
<td>$\gamma_s$</td>
<td>78 kN/m$^3$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>15 (°)</td>
</tr>
<tr>
<td>Number of agents</td>
<td>24</td>
</tr>
<tr>
<td>Number of iterations</td>
<td>400</td>
</tr>
</tbody>
</table>

After optimization process, the results of Table 3 are obtained. $k_h=0$ and $k_v=0$ are related to the design of the cantilever retaining wall under the static loading condition. The results of Table 3 are graphically shown in Fig. 4. Based on this figure, by increasing $k_h$ a more vigorous cantilever retaining wall is needed. But by increasing $k_v$, the inverse of this state becomes apparent. This behavior is predictable by considering Eq. (4). Thus in the design of the cantilever retaining walls in the prevalent conditions, ignoring $k_v$ is acceptable.

Table 3. The results obtained for optimum design of the cantilever retaining wall

<table>
<thead>
<tr>
<th>Variable</th>
<th>$k_h=0.0$</th>
<th>$k_h=0.1$</th>
<th>$k_h=0.2$</th>
<th>$k_v=0.0$</th>
<th>$k_v=0.05$</th>
<th>$k_v=0.1$</th>
<th>$k_v=0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>0.413</td>
<td>0.515</td>
<td>0.523</td>
<td>0.498</td>
<td>0.600</td>
<td>0.575</td>
<td>0.412</td>
</tr>
<tr>
<td>$T_2$</td>
<td>0.600</td>
<td>0.600</td>
<td>0.600</td>
<td>0.600</td>
<td>0.600</td>
<td>0.600</td>
<td>0.879</td>
</tr>
<tr>
<td>$T_3$</td>
<td>0.663</td>
<td>0.686</td>
<td>0.707</td>
<td>0.687</td>
<td>0.755</td>
<td>0.733</td>
<td>0.946</td>
</tr>
<tr>
<td>$T_4$</td>
<td>2.053</td>
<td>2.415</td>
<td>2.301</td>
<td>2.284</td>
<td>2.707</td>
<td>2.644</td>
<td>2.182</td>
</tr>
<tr>
<td>$T_6$</td>
<td>0.300</td>
<td>0.300</td>
<td>0.300</td>
<td>0.300</td>
<td>0.305</td>
<td>0.300</td>
<td>0.518</td>
</tr>
<tr>
<td>$T_7$</td>
<td>0.201</td>
<td>0.200</td>
<td>0.200</td>
<td>0.200</td>
<td>0.200</td>
<td>0.201</td>
<td>0.719</td>
</tr>
<tr>
<td>Best result</td>
<td>27.621</td>
<td>31.404</td>
<td>30.908</td>
<td>30.050</td>
<td>36.123</td>
<td>34.958</td>
<td>32.216</td>
</tr>
</tbody>
</table>

Fig. 4. Effect of $k_h$ and $k_v$ on the goal function

Figure 5 shows the convergence curve for obtaining the optimum solution. The utilized meta-heuristic methods are RO, CSS [14] and PSO.
Now consider the following two definitions:

\[ \text{Stability capacity ratio} = \frac{\text{Allowable safety factor}}{\text{Existing safety factor}} \]  

\[ \text{Shear capacity ratio} = \frac{\text{Existing shear force}}{\text{Allowable shear force}} \]

By these definitions, the optimum design of the cantilever retaining wall can be investigated in more detail. Table 4 provides these details and Fig. 6 is a graphical example of this point of view.

Table 4. Capacity assessment with CR meaning the capacity ratio

<table>
<thead>
<tr>
<th>Case</th>
<th>( k_h = 0.0 ), ( k_v = 0.0 )</th>
<th>( k_h = 0.1 ), ( k_v = 0.0 )</th>
<th>( k_h = 0.1 ), ( k_v = 0.05 )</th>
<th>( k_h = 0.1 ), ( k_v = 0.1 )</th>
<th>( k_h = 0.2 ), ( k_v = 0.0 )</th>
<th>( k_h = 0.2 ), ( k_v = 1.0 )</th>
<th>( k_h = 0.2 ), ( k_v = 0.2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear CR 1</td>
<td>23.47%</td>
<td>26.59%</td>
<td>26.49%</td>
<td>25.80%</td>
<td>29.41%</td>
<td>26.28%</td>
<td>25.40%</td>
</tr>
<tr>
<td>Shear CR 2</td>
<td>41.37%</td>
<td>49.86%</td>
<td>48.13%</td>
<td>46.15%</td>
<td>60.39%</td>
<td>57.27%</td>
<td>31.51%</td>
</tr>
<tr>
<td>Shear CR 3</td>
<td>86.04%</td>
<td>90.54%</td>
<td>94.22%</td>
<td>90.51%</td>
<td>100.13%</td>
<td>99.38%</td>
<td>48.66%</td>
</tr>
<tr>
<td>Shear CR 4</td>
<td>00.00%</td>
<td>00.00%</td>
<td>00.00%</td>
<td>00.00%</td>
<td>00.00%</td>
<td>00.00%</td>
<td>40.19%</td>
</tr>
<tr>
<td>Bearing CR</td>
<td>99.97%</td>
<td>99.98%</td>
<td>99.94%</td>
<td>99.86%</td>
<td>100.01%</td>
<td>100.00%</td>
<td>100.02%</td>
</tr>
<tr>
<td>Sliding CR</td>
<td>80.42%</td>
<td>86.65%</td>
<td>90.74%</td>
<td>92.49%</td>
<td>95.68%</td>
<td>99.06%</td>
<td>100.01%</td>
</tr>
<tr>
<td>Overturning CR</td>
<td>56.30%</td>
<td>54.03%</td>
<td>55.58%</td>
<td>55.00%</td>
<td>54.08%</td>
<td>53.80%</td>
<td>54.58%</td>
</tr>
</tbody>
</table>

Fig. 6. Capacity assessment, \( k_h = 0.2 \) and \( k_v = 0.1 \)
In a few cells of this table, the written capacity ratio is greater than 100%. As an example, the greatest numeral is 100.13%. However, this error is negligible, because the corresponding error of this numeral is 0.0013.

The most important controlling factor in the optimum design of cantilever retaining wall is the bearing capacity of the soil which is at the toe region. In all the cases, with a good approximation, the bearing capacity ratio is 100%. The shear capacity ratio at the toe region is the second controlling factor with average of 93% in all the cases except $k_h=0.2$ and $k_v=0.1$. Finally, the sliding capacity ratio is the last controlling factor which is increased from 80.42% to 100.01%. An important point in Table 4 is that, the shear capacity ratio at the heel region in all the cases except 7th case is 0.00%. This means the stress triangle has not entered into the heel region.

6. CONCLUDING REMARKS

The aim of the cantilever retaining wall optimization is to provide a design which not only satisfies the strength and stability constraints, but is also economical. In this paper, design of the cantilever retaining walls under various earthquake loading conditions is provided. These designs are performed by a new meta-heuristic optimization method called Ray Optimization. These designs reveal the following two results: Firstly, the bearing capacity of soil under toe region, sliding stability of cantilever retaining wall and the shear strength of the critical section in the toe region are the most important parameters in choosing the optimum design. Secondly unlike the expectation, the increase of the vertical component of the earthquake has a reverse effect on the design of the retaining walls.

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REFERENCES

APPENDIX A

Ray optimization method is used for maximization of $K_{ae}$ with respect to $\alpha$ and $t/T$ or $tc$.

### A.1. Magnitudes of $K_{ae}$, $tc$ and $\alpha$ for $k_v=0$

<table>
<thead>
<tr>
<th>$\Phi$ (°)</th>
<th>$\delta$ (°)</th>
<th>$k_v=0.0$</th>
<th>$k_v=0.1$</th>
<th>$k_v=0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td></td>
<td>$t_c$ (t)</td>
<td>$\alpha$ (°)</td>
<td>$K_{ae}$</td>
</tr>
<tr>
<td>-10</td>
<td>0.8366</td>
<td>61.5561</td>
<td>0.5779</td>
<td>0.4158</td>
</tr>
<tr>
<td>10</td>
<td>0.0471</td>
<td>51.0568</td>
<td>0.4467</td>
<td>0.4158</td>
</tr>
<tr>
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### A.2. Magnitudes of $K_{ae}$, $tc$ and $\alpha$ for $k_v=0.5 k_h$

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A. 3. Magnitudes of $K_{ae}$, $t_c$ and $\alpha$ for $k_h = k_v$

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Further information on retaining walls can be found in [15, 16, 17].

**APPENDIX B**

**B. 1. Two dimensional ray tracing.**

Consider Fig. App. B.1.

\[
t = -n \sqrt{1 - \frac{n_d^2}{n_i^2} \sin^2(\theta) + \frac{n_d}{n_i} (d - (d \cdot n) \cdot n)} \quad (B-1)
\]

Notice that $n$ and $d$ must be normalized vectors.

![Ray tracing in 2D space](image)

**B. 2. Three dimensional ray tracing.**

Define $i^*$ and $j^*$ as follows:

\[
i^* = \begin{cases} 
\frac{(n + d)}{\text{norm}(n + d)} & -1 \leq n \cdot d < -0.05 \\
\frac{-(n - d)}{\text{norm}(n - d)} & -0.05 \leq n \cdot d \leq 0.05 \\
\frac{(n - d)}{\text{norm}(n - d)} & 0.05 < n \cdot d \leq 1 
\end{cases}
\]

\[
j^* = \begin{cases} 
\frac{d}{\text{norm}(d)} & -1 \leq n \cdot d < -0.05 \\
\frac{(n - d)}{\text{norm}(n - d)} & -0.05 \leq n \cdot d \leq 0.05 \\
\frac{(n + d)}{\text{norm}(n + d)} & 0.05 < n \cdot d \leq 1 
\end{cases}
\]
Now define $n^*$ and $d^*$ as:

$$n^* = (1, 0)$$
$$d^* = (d_1^*, d_2^*, d_3^*)$$

Calculate $t^*$:

$$t^* = -n^* \sqrt{1 - \frac{n_d^2}{n_t^2} \sin^2(\theta)} + \frac{n_d}{n_t} (d^* - (d^* \cdot n^*) n^*)$$

Notice that $n^*$ and $d^*$ must be normalized vectors. $t^*$ is a two dimensional vector like $t^* = (t_1^*, t_2^*, t_3^*)$.

Finally calculate $t$:

$$t = t_1^* \cdot t_1^* + t_2^* \cdot t_2^*$$

**APPENDIX C**

**C. 1. Stability control**

All the loads acting on the cantilever retaining wall are shown in Fig. App. C.1.

Check for overturning:

$$FS_{overturning} = \frac{\sum M_r}{\sum M_o} \geq 1.5$$  \hfill (C-1)

$\sum M_r$: Sum of the moments of forces that tends to resist the overturning of the wall about C.

$\sum M_o$: Sum of the moments of forces that tends to overturn the wall about point C.
Check for sliding along the base:

\[ FS_{\text{sliding}} = \frac{P_{ae}}{\mu \sum W} \geq 1.5 \]  

(C-2)

\(P_{ae}\): Total seismic active thrust.

\[ \mu = \tan(\varphi) \]

Check for bearing capacity failure:

\[ FS_{\text{bearing capacity}} = \frac{q_u}{q_{\text{max}}} \geq 2 \]  

(C-3)

\(q_u\): Ultimate bearing capacity. It should be noticed that because of the existence of earthquake loading, the ultimate bearing capacity is increased by 33%.

\(q_{\text{max}}\): Maximum bearing pressure.

\[ q_{\text{max}} = \left\{ \begin{array}{ll} \sum W(1 + \frac{6e}{L}), & \text{for } e \leq \frac{L}{6} \\ \frac{BL}{2} \sum W, & \text{for } e > \frac{L}{6} \end{array} \right. \]  

(C-4)

where \(L\) is the footing length, \(B\) is the footing width which is equal to 1m, \(x\) is the length of the lever of the force about point C, and \(M_{ae}\) is the total moment of the seismic active thrust about point C.

**C. 2. Strength control**

The load combination is defined as the following, Ref. [13]:

\[ M_u = 1.2M_D + 1.4M_E \]

\[ V_u = 1.2V_D + 1.4M_E \]  

(C-5)

\(M_u\) and \(V_u\) are the ultimate moment and shear at the critical sections. The critical sections of the moment are shown in Fig. (3). The critical sections of the shear are at a distance \(d\) (effective depth) from the face of moment critical sections.

Check the shear capacity:

\[ \frac{V_u}{\phi_v V_n} \leq 1 \]  

(C-6)

\[ V_n = \frac{\sqrt{f_c}}{6}.B.d \]  

(C-7)

\(B\) is the length of shear critical section based on (mm), which is equal to 1000mm, \(\phi_v\) is equals 0.75, \(d\) is the effective depth, (mm) and \(f_c\) is the concrete strength (MPa).

Determining the required rebar:

Based on \(M_u\) (N.mm) in each critical cross section, the required area cross section of rebar, \(A_s\) based on (mm), is calculated. The minimum steel ratio is 0.0018.
\[ m = \frac{f_y}{0.85 f_c} \]
\[ R_n = \frac{M_u}{f_{y,B} B d^2} \]  \hspace{1cm} (C-8)
\[ A_s = \frac{B d}{m} \left[ 1 - \sqrt{1 - \frac{2 m R_n}{f_y}} \right] \]

\( \phi_b \) is equal to 0.9 and \( f_y \) is the rebar yield stress in MPa.