# APPLICATION OF KRONECKER PRODUCT TO THE ANALYSIS OF MODIFIED REGULAR STRUCTURES[*]

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**Abstract–** In this paper, structures transformable to regular forms are studied. Here, two cases are investigated. In the first case, the effect of different boundary conditions on these structures are explored, and in the second case the effect of adding or removing members and nodes are studied. In some structures the graph model is regular and different boundary conditions change the corresponding block matrices into non-regular ones. In some other structures the addition or removal of nodes and/or members changes the structure into a regular one. Here an efficient method is presented for dealing with the above-mentioned irregularities.

The main idea steams from the fact that on the one hand there exist simple relationships for finding the inverse of some block matrices related to regular structures, and on the other hand we want to find out how to obtain the inverse of matrices corresponding to structures which become regular by the addition or removal of some members and/or nodes.

One of the applications of the present method is related to the finite difference (FD) method for the analysis of plates with some irregularities in their boundary or having different support conditions.

**Keywords –** Kronecker product, graphs, regular structures, plates, finite difference method

## 1. INTRODUCTION

Most of the research results presented in the past on regular graphs belong to the eigensolution of their matrices. A structure is called *regular* if it can be represented as a product graph [1-3]. In these methods, the calculations can be performed using the theorems of Refs. [4-5]. When a node (or a set of nodes) and the connected members are added to regular structures, the corresponding matrices can still be considered as the sum of some Kroneker products, the problem can be solved provided the symmetry of the structure is maintained [6]. In general, the regular structures are studied from the point of view of having repetitive units or symmetry [7-9]. In reference [10] using *QZ* decomposition, a method has been presented to calculate the inverse of those matrices which are in the form of the sum of two Kronecker products. In the general case, one may come to structures which cannot be considered regular with regard to their graph models, however, by adding or deleting a few nodes and/or members these can be changed into the previously solved forms. Here, it will be shown that this addition or deletion need not preserve the symmetry of the structure.

The problem of inverting matrices associated with modified regular structures can be treated more efficiently using an approach that takes into account the solutions of well-formed matrices of the main regular structures. Here a well-formed matrix is defined as a matrix with a canonical form for which the

inversion is carried out using much simpler formulations. Using such an approach one may consider various types of modifications. For example, different boundary conditions can be treated using this method. Also, in the analysis of some structures such as plates with irregular boundaries using the FD method, the present approach can be efficiently used employing the solution of the plate with regular configuration.

In what follows, first the method for calculating the inverse of block matrices is discussed and then a method is presented for finding the inverse of those matrices which are transformable to regular ones. Finally, the application of this method is illustrated through some examples.

## 2. INVERSION OF BLOCK MATRICES

First, it should be noted that in matrix algebra the inverse of a block matrix can be obtained in terms of the inverse of its blocks by a special formulation. However, such an operation requires the inversion of the blocks involved. Here we will observe that considering the eigenvalues and eigenvectors of block matrices [4-5], such calculation can be simplified.

Suppose the form of the matrix to be investigated is as follows:

$$\mathrm{M}_{mn} = A_1 \otimes B_1 + A_2 \otimes B_2 \tag{1}$$

where $n$ is the dimension of the matrices $A_1$ and $A_2$, and $m$ is the dimension of the matrices $B_1$ and $B_2$. The sign $\otimes$ shows the Kronecker product of two matrices and it is defined in the Appendix. We consider two cases. In the first case, $A_1 A_2 = A_2 A_1$, then using the eigenvalues the inverse of $M$ can easily be obtained. In the second case, $A_1 A_2 \neq A_2 A_1$ and we should use $QZ$ factorization.

In the first case, the two matrices $A_1$ and $A_2$ commute with respect to multiplication, and one can find the eigenvalues using the following relationship:

$$\lambda_M = \bigcup_{i=1}^{n} \mathrm{eig}(M_i); M_i = \lambda_i(A_1)B_1 + \lambda_i(A_2)B_2 \tag{2}$$

It is obvious that if $V$ is the matrix containing the eigenvectors and $D$ is a diagonal matrix containing the eigenvalues of a symmetric matrix $M$, then we will have $M = VDV^t$. Since the eigenvalues of $M^{-1}$ are the inverse of those of $M$ and the eigenvectors are identical, therefore

$$M^{-1} = VD^{-1}V^t = V\begin{bmatrix} 1/\lambda_1 & & & & 0 \\ & 1/\lambda_2 & & & \\ & & . & & \\ & & & . & \\ 0 & & & & 1/\lambda_{m*n} \end{bmatrix} V^t \tag{3}$$

Where $D^{-1}$ can easily be obtained by inverting the diagonal entries of $D$. The eigenvector of such a matrix will be in the form of $u \otimes v$ in which $u$ is a vector that diagonalizes both matrices $A_1$ and $A_2$ simultaneously and $v$ which is an eigenvector of $M_i = \lambda_i(A_1)B_1 + \lambda_i(A_2)B_2$ as discussed in [11].

In the second case, if $A_1$ and $A_2$ do not commute with respect to multiplication, then $QZ$ decomposition should be used. This decomposition is introduced in [7] and here only the inverting process is re-introduced.

In this case, consider $y = Mx$, where $M$ has the form of Eq. (1). Natural approach will lead to $x = M^{-1}y$. Here, one can use $QZ$ decomposition [12]. However, instead of inversion, we consider appropriate transformations to make $M$ a diagonal matrix and inversion can then be achieved by inverting the diagonal

entries. For this purpose, decomposition should be performed such that instead of *T*, we end up with a diagonal *D*. We use $QZ$ decomposition as

$$\begin{cases} T_\text{A} = QAZ \\ T_\text{B} = QBZ \end{cases} \quad \alpha = \text{diag}(T_\text{A}) \ , \ \beta = \text{diag}(T_\text{B}) \tag{4}$$

α and β are the entries on the main diagonal of $T_\text{A}$ and $T_\text{B}$, respectively.

Substitute $U = K^{-1}$ with the following:

$$K(:,\text{i}) = \begin{cases} AV(:,\text{i}) & |\alpha_\text{i}| \geq |\beta_\text{i}| \\ BV(:,\text{i}) & \text{elsewhere} \end{cases} \quad (V = Z^\text{t}) \tag{5}$$

then unlike the previous case we have $V = Z^\text{t}$, but $U \neq Q^\text{t}$ and $UAV$ is a diagonal matrix:

$$y = (A_1 \otimes B_1 + A_2 \otimes B_2)x \tag{6}$$

Considering

$$U_\text{A} A_1 V_\text{A} = D_1 \ , \ U_\text{A} A_2 V_\text{A} = D_2 \ , \ U_\text{B} B_1 V_\text{B} = D_3 \ , \ U_\text{B} B_2 V_\text{B} = D_4 \tag{7}$$

We have

$$(U_\text{A} \otimes U_\text{B})y = (U_\text{A} \otimes U_\text{B})(A_1 \otimes B_1 + A_2 \otimes B_2)x \tag{8}$$

Substituting

$$x = (V_\text{A} \otimes V_\text{B})\bar{x}, \quad \bar{y} = (U_\text{A} \otimes U_\text{B})y \tag{9}$$

We have

$$\bar{y} = (D_1 \otimes D_3 + D_2 \otimes D_4)\bar{x} \tag{10}$$

Having $y$, $\bar{y}$ can be calculated and since the matrix in prentices is diagonal, $\bar{x}$ and then $x$ can be calculated. It should be noted that in using these transformations, the calculations are performed on $A_1$, $A_2$, $B_1$ and $B_2$ having dimensions similar to that of the repetitive blocks. Thus the amount of calculations is reduced considerably.

## 3. PROPOSED METHOD

Most of the research results presented in the past were concentrated on structural forms having support conditions for which the structural matrices could be expressed as the sum of some Kronecker products. In such cases using the corresponding theorems one can simplify the calculations using the block matrices. In general a structure can have supports leading to non-regular forms which make these calculations impossible. The structure can also have a geometry which can be transformed into regular models by adding some members and nodes. The main aim of this paper is to study such cases and, as an example, the calculations will be performed on the matrices corresponding to FD solutions.

### a) The effect of different boundary conditions

In order to clarify the problem, suppose that we want to calculate the maximum deflection of the plate shown in Fig. 1. This plate is uniformly loaded and it is simply supported in three edges and clamed in the other edge. This problem is solved in Ref. [11] for the case where the plate is simply supported in its four sides.

The governing equation for this problem is as follows [13]:

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \quad ; \quad D = \frac{Et^3}{12(1-\upsilon^2)} \tag{11}$$

If the length of the subdivisions in both directions is taken as h, for a typical joint (*i,j*), we will have

$$\begin{aligned}
&20w(i, j) - 8\big[w(i, j-1) + w(i+1, j) + w(i, j+1) + w(i-1, j)\big] \\
&+ 2\big[w(i-1, j-1) + w(i+1, j-1) + w(i+1, j+1) + w(i-1, j+1)\big] \\
&+ \big[w(i, j-2) + w(i+2, j) + w(i, j+2) + w(i-2, j)\big] = \frac{ph^4}{D}
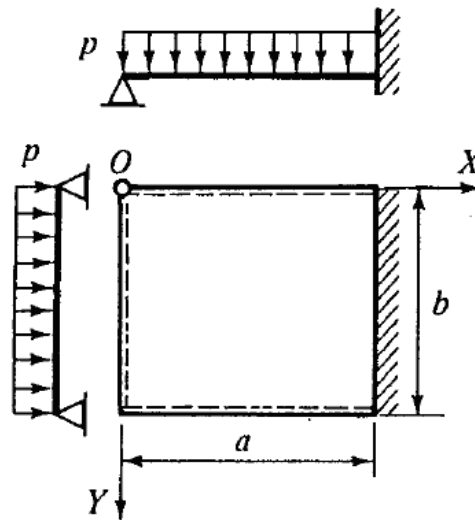\end{aligned} \tag{12}$$



Fig. 1. A plate with simple supports in three edges and clapped in one edge under a uniform loading

In this example, the central FD operator is used. Any other FD operator or numerical method can be employed, however, the matrix corresponding to its graph model should fulfill the commutativity condition $A_1 A_2 = A_2 A_1$ . Since in the central FD operator for the second-order partial differential equation every node is connected to its 4 adjacent nodes at the top, bottom, left and right, the graph of the model is in the form of the Cartesian product of two paths. As it is shown in Refs. [4, 5], since for this graph the above commutativity condition holds, it can be decomposed. As an example, the matrix corresponding to the fourth-order partial differential equation that uses Eq. (12), satisfies the commutativity condition. Therefore the present method is applicable only when firstly the corresponding model is regular, and secondly the nodal numbering is performed such that the decomposibility condition is not violated.

It should be noted that the nodal numbering is performed such that the nodes corresponding to the clamed supports and the corresponding nodes in the other side of the plate are first numbered, followed by the numbering of the remaining nodes. As an example, if we consider the numbers of subdivisions in the X and Y directions as 7 and 6, respectively, the nodal numbering should be performed as illustrated in Fig. 2. In this way writing the FD equations and imposing the boundary conditions, we will obtain the following matrix:

$$[C]\{w\} = -\{p\}\frac{h^4}{D} \quad \Rightarrow \quad \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = -\frac{h^4}{D}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} \quad ; \quad C_{21} = C_{12}^t \tag{13}$$

where the decomposed submatrices are as follows:

$$C_{11} = F_5(A_2, B_2, A_2 + I_2, I_2) \quad ; \quad A_2 = \begin{bmatrix} 20 & 0 \\ 0 & 18 \end{bmatrix} \quad ; \quad B_2 = -8 \times I_2$$

$$C_{22} = F_5(A_4, B_4, A_4 + I_4, I_4) \quad ; \quad A_4 = F_4(19, -8, 19, 1) \quad ; \quad B_4 = F_4(-8, 2, -8) \tag{14}$$

$$C_{12} = F_5(A_{4 \times 2}, B_{4 \times 2}, A_{4 \times 2}) \quad ; \quad A_{4 \times 2} = \begin{bmatrix} -8 & 1 & 0 & 0 \\ 0 & 0 & 1 & -8 \end{bmatrix} \quad ; \quad B_{4 \times 2} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$
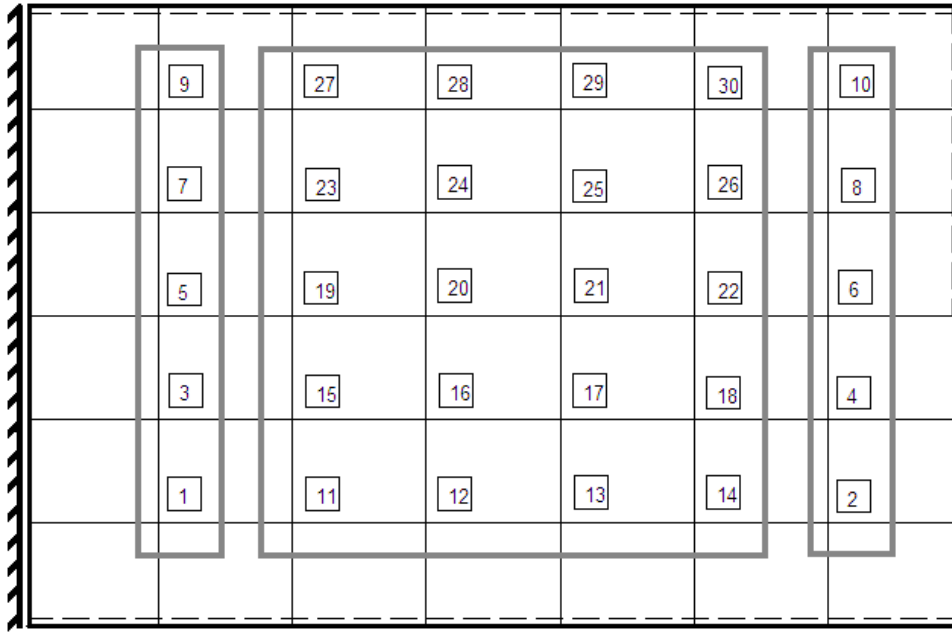


Fig. 2. Nodal numbering of the plate for the FD method

Where $I$ is a unit matrix and

$$F_m(A, B, C, D) = \begin{bmatrix} A & B & D & & & & 0 \\ B & C & B & . & & & \\ D & B & . & . & . & & \\ & . & . & . & B & D \\ & & . & B & C & B \\ 0 & & & D & B & A \end{bmatrix}_m \tag{15}$$

if the 4$^{th}$ argument of F is not present, then the above matrix becomes a block tri-diagonal matrix.

Considering the different boundary conditions and using the above-mentioned nodal numbering, one can observe repetitive block forms in all the submatrices (even in the submatrix $C_{21} = C_{12}^t$ which is a rectangular matrix)

Therefore, solving Eq. (13) in block form we will have:

$$\begin{cases} \{w_2\} = -\dfrac{h^4}{D} C_{22}^{-1} [\{p_2\} + C_{21}\{w_1\}] \\ \{w_1\} = -\dfrac{h^4}{D} [C_{11} - C_{12}C_{22}^{-1}C_{21}]^{-1} [\{p_1\} - C_{12}C_{22}^{-1}\{p_2\}] \end{cases} \tag{16}$$

It can be seen that in these relationships we do not need to find the inverse of $C_{11} - C_{12}C_{22}^{-1}C_{21}$ and $C_{22}$. The important point is that, as can be seen the inverse of $C_{11}$ and $C_{22}$ can easily be found and we need

only to invert $C_{11} - C_{12}C_{22}^{-1}C_{21}$ which is a matrix with identical dimensions to that of $C_{11}$. Since in most of the numerical methods like FD the number of subdivisions is generally high, the dimension of the matrix $C_{11}$ is less than that of $C_{22}$, and from the second row, the matrix $w_2$ should be calculated, otherwise we should first calculate $w_1$.

Now we consider the inversion of $C_{11}$ and $C_{22}$ matrices. These matrices are block five-diagonal matrices. As an example, the inverse of $C_{22}$ can be expressed as

$$C_{22} = F(A, B, A+I, I) = I \otimes A + F(0,1,0) \otimes B + F(0,0,1,1) * I = \sum_{i=1}^{3} A_i \otimes B_i \qquad (17)$$

Since we have three Kroneker products, $QZ$ transformation of inversion cannot be used. However, since $A_i A_j = A_j A_i$ we first use Eq. (2) and obtain

$$\lambda_{C_{22}} = \bigcup_{i=1}^{5} eig(M_i); M_i = A + \lambda_i (F(0,1,0))B + \lambda_i (F(0,0,1,1))I \qquad (18)$$

The vector $u$ which diagonalizes the three matrices $I_5$, $F(0,1,0)$ and $F(0,0,1,1)$ simultaneously will then be calculated followed by the vector $v$, i.e. the eigenvector of $M_i = A + \lambda_i (F(0,1,0))B + \lambda_i (F(0,0,1,1))I$. The columns of the matrix $V$ are equal to $u \otimes v$. In this way, using Eq. (3), the $C_{22}^{-1}$ will be obtained.

In relation to $C_{22}$ it is important to note that this matrix is exactly the same as the matrix we had for the plate with 4 edges being simply supported. In fact, the role of different support conditions is reflected in $C_{11}$ and $C_{21} = C_{12}^t$.

For inverting the submatrix $C_{11}$ a similar calculation can be carried out. If we only want to calculate the moments, we have to adopt a similar process. In this case, we will have block tri-diagonal matrix which can be expressed as the sum of two Kronecker products and the process of calculation will not be different.

### b) Structures transformable to regular forms

In the following we study the analysis of plates. In general, one needs to calculate the moments and deflection of the plates where its geometry can be changed into regular figures by adding or deleting some parts. Then using the FD method and after writing the corresponding equations for the regular plate, the results of the main plate are obtained.

As an example, we want to calculate the moments and deflection of the plate in Fig. 3a. This plate is pinned at its boundary nodes. For solution FD method is employed. As can be seen, for simplicity and because of the irregularity of the plate, nine nodes are selected. Obviously for more irregularity the distance between the nodes should be reduced by increasing the number of subdivisions.

First we transform the plate into a complete plate as shown in Fig. 3b, and with new numbering the inverse of the matrix of the FD equations of this plate can be calculated.

The governing relationships for this problem are as follows:

$$[C_{(b)}]\{M\} = -\{P\}a^2 \quad \Rightarrow \quad \{M\} = -[H]\{P\}a^2 \quad ; \quad [H] = [C_{(b)}]^{-1} \qquad (19)$$

By partitioning this matrix we have

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = - \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} P_1 a^2 \\ P_2 a^2 \end{Bmatrix} \qquad (20)$$

which can be expressed as

$$\begin{Bmatrix} M_1' \\ 0 \end{Bmatrix} = -\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{Bmatrix} P_1 a^2 \\ P_2' a^2 \end{Bmatrix} \qquad (21)$$
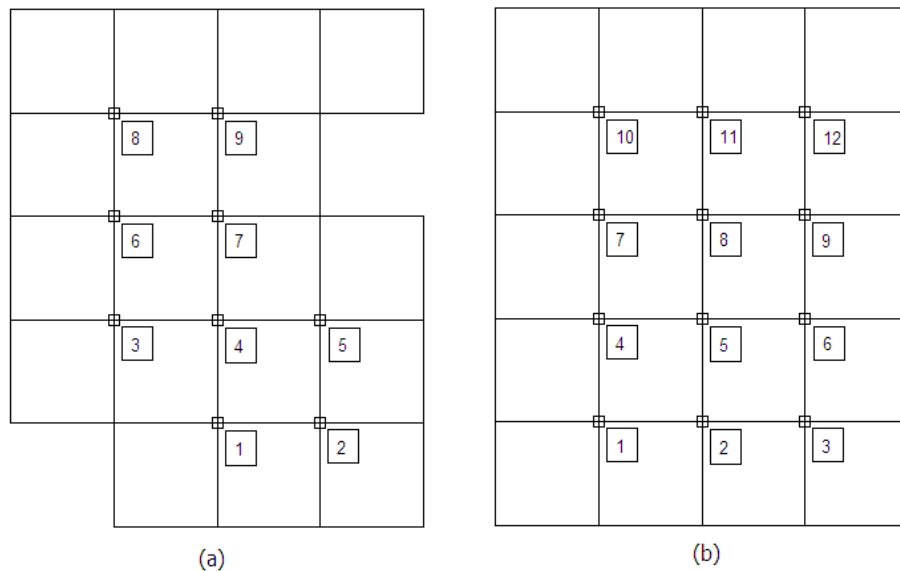


Fig. 3. Completion of (a) as (b) for having a regular block matrix

The governing relationships for this problem are as follows:

$$[C_{(b)}]\{M\} = -\{P\}a^2 \quad \Rightarrow \quad \{M\} = -[H]\{P\}a^2 \quad ; \quad [H] = [C_{(b)}]^{-1} \qquad (19)$$

By partitioning this matrix we have

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = -\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{Bmatrix} P_1 a^2 \\ P_2 a^2 \end{Bmatrix} \qquad (20)$$

which can be expressed as

$$\begin{Bmatrix} M_1' \\ 0 \end{Bmatrix} = -\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{Bmatrix} P_1 a^2 \\ P_2' a^2 \end{Bmatrix} \qquad (21)$$

Here, $H_{22}$ corresponds to the added nodes (3 nodes) and $H_{11}$ corresponds to the remaining nodes. Thus

$$M_1' = -[H_{11}P_1 + H_{12}P_2']a^2 \quad ; \quad P_2' = -H_{22}^{-1}H_{21}P_1 \qquad (22)$$

Combining these two equations leads to

$$M_1' = -[H_{11} - H_{12}H_{22}^{-1}H_{21}]P_1 a^2 \quad \Rightarrow \quad C_{(a)}^{-1} = H_{11} - H_{12}H_{22}^{-1}H_{21} \qquad (23)$$

Ultimately the deflection of the plate is obtained as

$$[C_{(a)}]\{W'\} = -\frac{M_1'}{D}a^2 \quad \Rightarrow \quad \{W'\} = -[H_{11} - H_{12}H_{22}^{-1}H_{21}]\frac{M_1'}{D}a^2 \qquad (24)$$

where $D = Et^3/12(1-v^2)$.

In example 1, it can be seen that the selected grid for discretization of the plate for FD analysis has certain extra parts compared with a standard graph product.

The point is that, in some plates it is easier to consider it in the form of a sector of a circle in place of considering it as a rectangular model. Example 2 investigates a plate using this approach.

In any case we should know that a rectangular or a sector of a circle can be inscribed in a circle or circumscribed, in both cases the calculations lead to the inversion of a matrix with dimension equal to the number of nodes added or deleted. As an example, in Fig. 4 for both irregular plates, in analysis by FD method, both cases of the plates inscribed in a circle or circumscribed by it are illustrated.
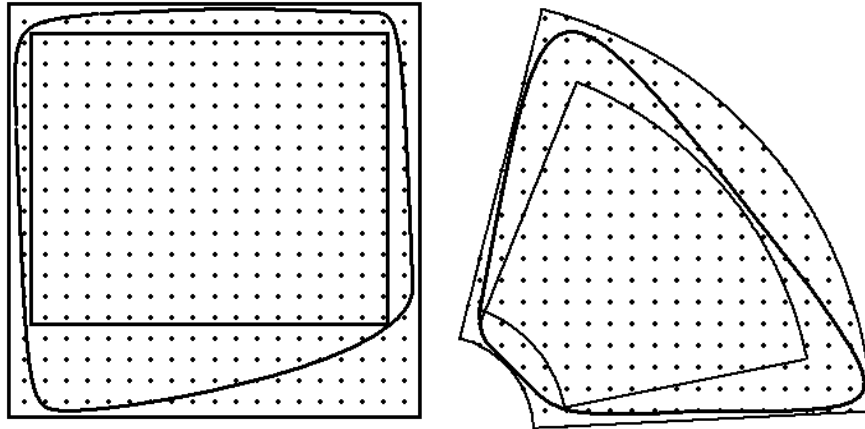
Fig. 4. Transformation of two non-regular plates to regular rectangular- and
sector of a circle-shaped plates

## 4. NUMERICAL EXAMPLES

**Example 1**: In this problem a plate descretized as shown in Fig. 5 for FD analysis, contains parts more that of a standard product graph. In the previous section we studied a plate smaller than its circumferential bigger plate, while here we create a regular product graph which is inside the plate. Partitioning the stiffness matrix of this plate into two parts, and having the inverse of the created product graph, the solution of the problem becomes feasible.
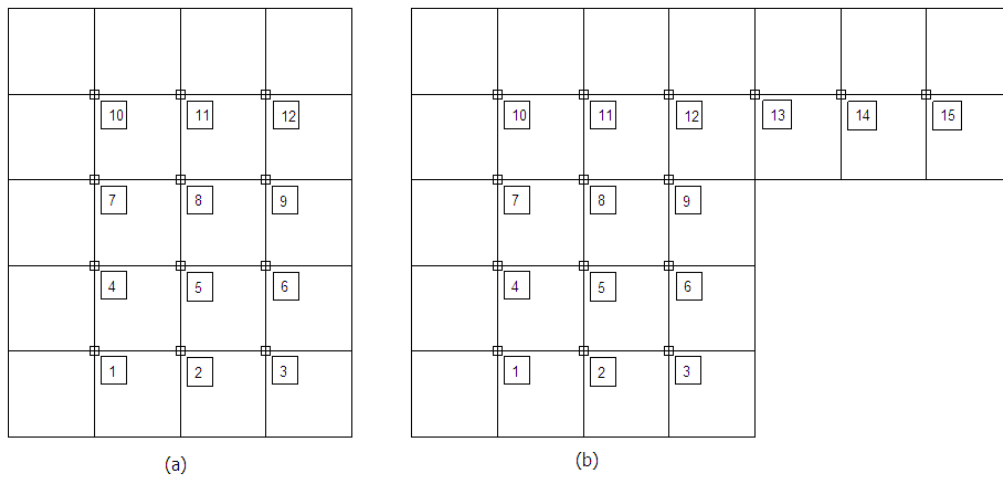
Fig. 5. Reduction of (b) to obtain (a) for having a regular block matrix

Here the equations are written for Fig. 5a and ultimately the results are obtained for Fig. 5b. The governing relationship here is as follows:

$$[C_{(a)}]\{M\} = -\{P\}a^2 \quad \Rightarrow \quad \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = - \begin{Bmatrix} P_1 a^2 \\ P_2 a^2 \end{Bmatrix}$$

Therefore

$$\begin{cases} \{M_1\} = C_{11}^{-1}[\{P_1\}a^2 - C_{12}\{M_2\}] \\ \{M_2\} = [C_{22} - C_{21}C_{11}^{-1}C_{12}]^{-1}[\{P_2\} - C_{21}C_{11}^{-1}\{P_1\}]a^2 \end{cases}$$

Substituting $M_2$ in $M_1$, and due to the ease of inverting $C_{11}$, it is enough to find the inverse of a matrix having dimension equal to that of $C_{22}$, i.e. we calculate $[C_{22} - C_{21}C_{11}^{-1}C_{12}]^{-1}$.

Now we calculate the deflection as follows:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = -\frac{1}{D} \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$$

Thus

$$\begin{cases} \{W_1\} = C_{11}^{-1}[-\dfrac{M_1}{D} - C_{12}\{W_2\}] \\ \{W_2\} = [C_{22} - C_{21}C_{11}^{-1}C_{12}]^{-1}[C_{21}C_{11}^{-1}\dfrac{M_1}{D} - \dfrac{M_2}{D}] \end{cases}$$

In this way the inverse of the same matrix $[C_{22} - C_{21}C_{11}^{-1}C_{12}]$ which resulted in the bending moments is obtained for calculating the deflections.

**Example 2**: Here, we study a plate which is convertible to the sector of a circle. As it can be seen from Fig. 6a, this plate can be converted into the quarter of a circle by adding some parts, and writing the finite difference equations in the polar coordinate system as developed in [14], it has become complete as shown in Fig. 6b. It should be noted that all the supports are pinned around the edges.



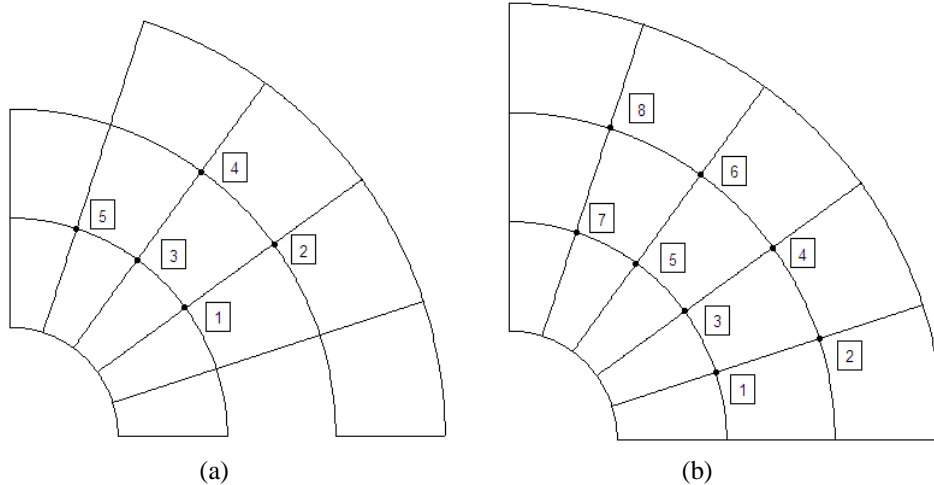(a)                                        (b)
Fig. 6. Completion of (a) to obtain (b) for having regular block matrix in polar coordinate system

The finite difference matrix for the plate with hole in Fig. 6a is a 5 by 5 non-rectangular one. Adding 3 nodes, the plate changes into a quarter of a circle having 8 nodes and the corresponding FD matrix take the form $F$ which can easily be inverted using Eq. (3).

$$C_{(b)} = F_4(A_2, B_2, A_2) = I_4 \otimes A_2 + F_4(0,1,0) \otimes B_2$$

$$A_2 = \begin{bmatrix} 2(1+\dfrac{25}{\pi^2}) & \dfrac{5}{4} \\ \dfrac{5}{6} & 2(1+\dfrac{100}{9\pi^2}) \end{bmatrix} \quad ; \quad B_2 = \begin{bmatrix} \dfrac{25}{\pi^2} & 0 \\ 0 & \dfrac{100}{9\pi^2} \end{bmatrix}$$

In this way, using the relationships of Example 1, one needs to invert only a matrix of dimension 3 corresponding to the added nodes.

For some other applications of graph theory in structural mechanics the reader may refer to Refs. [15-20].

## 5. CONCLUDING REMARKS

The method presented in this paper extends the applications of the previous methods developed for the analysis of regular structures based on the stiffness method and some concepts from graph products. In some of the examples previously investigated, adding or removing some nodes and/or members changes the models into non-regular ones, or similarly the use of different support conditions may alter the repetitive nature of the corresponding block matrices. In such cases, first the matrices are partitioned in such a way that the effect of the support conditions and the remaining part of the structure are separated, and the analysis is performed using the regularity property. Thus the support conditions do not need to be regular and different supports can be present in the structure.

In this paper, some relationships are developed for the inverse of a set of block matrices such that one can easily find the inverse of the matrix of a structure when it is transformable into a regular one. In general, two cases may arise. In the first case some nodes and members are added to regular structures, and in the second case some nodes and members are removed from a regular structure. The main idea is related to the modifications of structures. The present method results in a considerable reduction in computational time due to the decomposition of large matrices into smaller ones and using special methods for their inversion. As an application, the analysis of plates using FD method is performed, where moments and deflections of plates are calculated by transforming their configurations into regular models.

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## APPENDIX

### *A.1 Definitions from Graph Theory*

A *graph S(N,M)* consists of a set of elements, $N(S)$, called *nodes* and a set of elements, $M(S)$, called *edges (members)*, together with a relation of incidence which associates two distinct nodes with each edge, known as its *ends*. A *subgraph $S_i$* of a graph $S$ is a graph for which $N(S_i) \subseteq N(S)$ and $M(S_i) \subseteq M(S)$, and each edge of $S_i$ has the same ends as in $S$. A *path graph P* is a simple connected graph with $N(P) = M(P)+1$ that can be drawn in a way that all of its nodes and edges lie on a single straight line. A path graph with $n$ nodes is denoted as $P_n$. A *cycle graph* is a simple connected graph with $N(C) = M(C)$ that can be drawn in a manner that all of its nodes and edges lie on a circle. A cycle graph with $n$ nodes is denoted as $C_n$. For further definitions and some application the reader may refer to Refs [15, 16].

### *A.2 Kronecker Product*

The *Kronecker product* of two matrices *A* and *B*, is the matrix we get by replacing the *ij*-th entry of *A* by $a_{ij}B$, for all *i* and *j*.

As an example,

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b & a & b \\ c & d & c & d \\ a & b & 0 & 0 \\ c & d & 0 & 0 \end{bmatrix} \tag{A-1}$$

where entry 1 in the first matrix has been replaced by a complete copy of the second matrix.

The Kronecker product has the property that if *B*, *C*, *D*, and *E* are four matrices, such that *BD* and *CE* exists, then

$$(B \otimes C)(D \otimes E) = BD \otimes CE \qquad \text{(A-2)}$$

Thus, if *u* and *v* are vectors of the correct dimensions, then

$$(B \otimes C)(u \otimes v) = Bu \otimes Cv \qquad \text{(A-3)}$$

If *u* and *v* are eigenvectors of *B* and *C*, with eigenvalues λ and μ, respectively, then

$$Bu \otimes Cv = \lambda\mu u \otimes v \qquad \text{(A-4)}$$

Whence *u*⊗*v* is an eigenvector of *B*⊗*C* with eigenvalue λμ.