IJST, Transactions of Civil Engineering, Vol. 39, No. C1, pp 1-19
Printed in The Islamic Republic of Iran, 2015

FINITE ELEMENT ANALYSIS USING MIXED FORCE-DISPLACEMENT METHOD VIA SINGULAR VALUE DECOMPOSITION*

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Abstract– The present approach is a combination of the force method and displacement approach to achieve the analysis using the substructuring technique. In this method, the inverse of the stiffness matrices of the substructures are constructed for the formation of the flexibility matrices. This part of the solution is equivalent to the stiffness approach. In the subsequent stage, the results of the analysis are assembled using the singular value decomposition (SVD) and the solution for the entire structure is obtained. In fact, for assembling the structure, we need the flexibility matrices of the substructures which are obtained by the stiffness method.

In this paper, a mixed force-displacement method is applied to finite element models for increasing the speed of their solution. Each substructure is analyzed independently by singular value decomposition of the corresponding equilibrium matrix. Methods are then utilized for transforming the substructures into regular forms whenever it is possible. The application of this method in finite element models with different substructures improves the process of analysis, and makes the use of the existing solution techniques possible for regular systems.

Keywords– Mixed force-displacement method, substructuring, singular value decomposition of equilibrium matrix, regular forms, finite element analysis

1. INTRODUCTION

Substructuring techniques are developed for static and dynamic analyses of large-scale structures [1-6]. The main idea comes back to concepts that are encountered in numerical solution of partial differential equations [7-8]. Dividing the model of a large-scale structure into smaller substructures, one can find the solution of the structure based linear combination of the solutions of its substructures [9-10]. Since the independent analysis of the substructures is much simpler than the solution of the entire structure, utilizing this method reduces the cost of computation to a great extent [11-13]. Substructuring technique is a powerful tool for those structures which become regular by addition or removal of some elements. This method is a suitable means for structures with repeated models [14-16]. Recently an efficient approach has been developed for linear and non-linear analysis of structure with tubular elements using the substructuring technique [17].

A structure is called *regular* if its model contains a special pattern. As an example, a circular structure consists of repeating units which are all identical, or a translational model has a repeating unit with identical repeating units, where the first and last units are non-identical. These models lead to matrices of special patterns which are called *canonical forms*. These matrices are often block matrices

*Received by the editors September 27, 2013; Accepted May 26, 2014.
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with blocks being positioned in a special manner. The eigensolution of these matrices and solution of the corresponding structures can be performed much easier as discussed in [18-20].

For the analysis of the structure using this method, first the equilibrium matrix of the substructures should be constructed. Using its singular value decomposition and inverting their stiffness matrices one can obtain the solution of the entire structure. Due to the block diagonal form of the equilibrium matrix of the substructures, the singular value decomposition of this matrix can easily be performed, since many of these matrices are identity matrices. This method can also transform some of the structural forms to canonical forms. If a substructure contains a regular form, one can easily perform its analysis using the existing efficient approaches [16-20].

Using the present method, having the inverse of the stiffness matrices of the substructures, the results can be obtained for the entire structure. This can be achieved by decomposition of small blocks of the equilibrium matrices. In this paper the term "decomposition" refers to SVD when it is used in conjunction with matrices, and it refers to sub-structuring when a structure or its model is involved.

In fact, the present approach is a combination of the force method and displacement approach to achieve the analysis using the substructure technique. In this method, the inverse of the stiffness matrices of the substructures are constructed for the formation of the flexibility matrices. This part of the solution is equivalent to the stiffness approach. In the subsequent stage the results of the analysis are assembled using the SVD and the solution for the entire structure is obtained. Indeed, for assembling the structure we need the flexibility matrices of the substructures which are obtained by the stiffness method. Therefore, the present method can be viewed as a mixed force-displacement method.

In this paper, first the method of formation of equilibrium matrix is described. The singular value decomposition is presented. Then the nodal displacements of the structure are calculated using the method of Ref. [21] and the analysis of the substructures. In the subsequent section a method is developed for modifying the substructures based on concepts of Section 2. In Section 6 three practical structures are analyzed and the computational times are compared to those of a direct method. In each example we have tried to simulate the real conditions for the structures. Section 7 concludes the paper.

2. SUBSTRUCTURES SEPARATION AND EQUILIBRIUM MATRIX FORMATION

In this section first the equilibrium matrix of the substructures is introduced and then the singular value decomposition of this is employed for the analysis of the structure.

First, we consider two substructures m and n as shown in Fig. 1 which are connected to each other at two nodes.

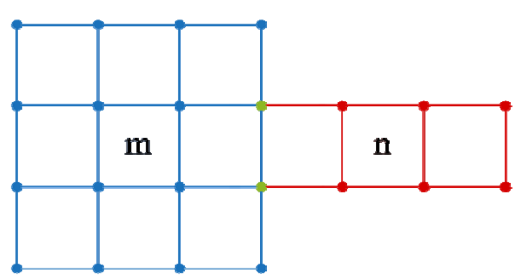


Fig. 1. A structure composed of two substructures m and n

The nodal DOFs of the substructures m and n depending on whether they share any commonalities with both substructures or not, can be decomposed in the form shown in Fig. 2.

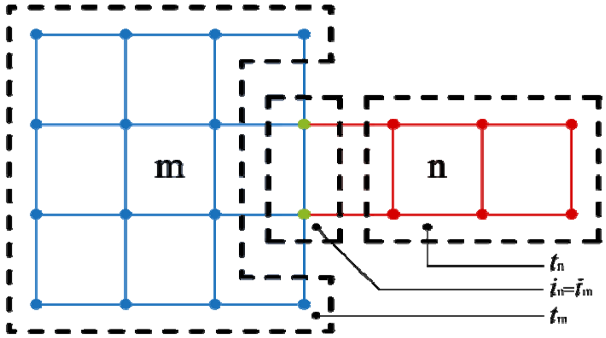


Fig. 2. Decomposition of the nodal DOFs of the structure

In what follows the DOFs corresponding to the substructures m and n will be identified by N_m and N_n, respectively, as shown in Fig. 3.

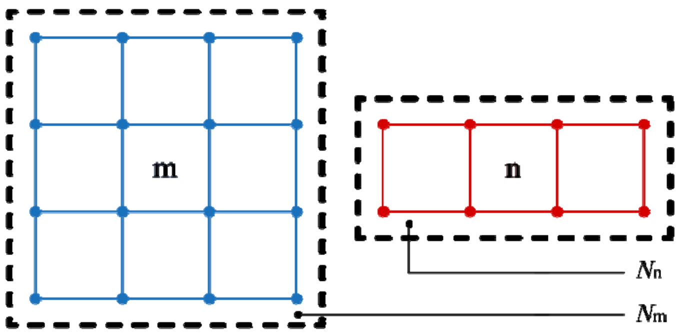


Fig. 3. Notations used for DOFs of the substructures

Here N_m and N_n are subsets of DOFs corresponding to substructures m and n, respectively when these DOFs are independent of each other. Also i_m and i_n are subsets of the DOFs corresponding to substructures m and n at the interface of these substructures, respectively. According to these definitions, the following relationships hold for the parameters defined in Figs. 2 and 3:

$$\begin{aligned} t_m &= N_m - N_n \\ t_n &= N_n - N_m \\ i_m &= i_n = N_m \cap N_n \end{aligned} \tag{1}$$

According to the above classification of the structural nodes, the equilibrium matrix for the substructures in the global coordinate system of the structure will be defined as Eqs. (2 and 3):

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_{t_m \times t_m} & & \\ & \mathbf{A}_1 & \\ & & \mathbf{I}_{t_n \times t_n} \end{bmatrix} \tag{2}$$

$$\mathbf{A}_1 = \left[\mathbf{I}_{i_m \times i_m} \mid \mathbf{I}_{i_n \times i_n} \right] \tag{3}$$

I is an identity matrix and matrix $\mathbf{A}_1$ is the result of imposing the equilibrium condition on the common nodes the two substructures. Therefore the dimension of the matrix $\mathbf{A}_1$ is dependent only on the DOFs of the common nodes of the two substructures. This dimension is often much smaller than the total DOFs of the structure.

The equilibrium matrix, in general, can be constructed by writing equilibrium equations among the DOFs of the substructures shown in Fig. 3. However, this matrix can be transformed into the form shown in Eq. (2) with a suitable ordering.

In the following a method is presented for the formation of the equilibrium matrices of the substructures, however, one does not need to use this approach and other method can be utilized. The use of the suggested method increases the capability of the analysis.

a) Joining two substructures with connecting elements

In general one can assume that the two substructures are connected to each other by some elements (as some connecting elements and not an independent substructure). This type of element is common in finite element modeling, such as "contact element" in tunnel models. As an example, suppose two substructures m and n are connected to each other by a substructure r, as shown in Fig. 4. In this case the DOFs of the substructures are decomposed as shown in Figs. 4 and 5.

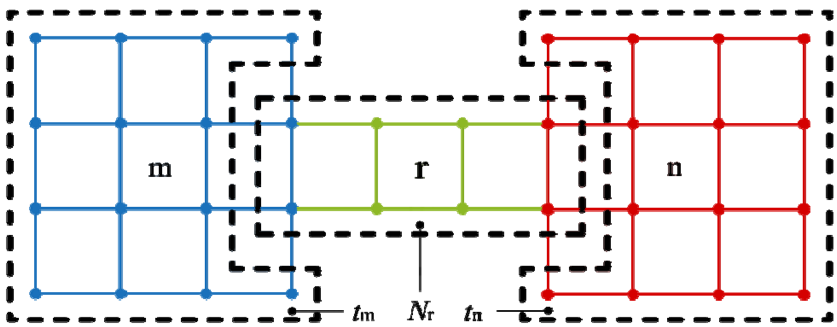


Fig. 4. Decomposition of the nodal DOFs of the structure and the connecting elements

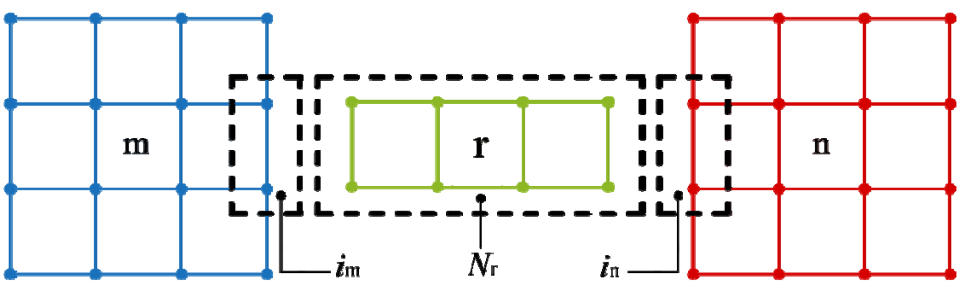


Fig. 5. Classification of the DOFs of the substructures

N_r is the set of DOFs for the connecting elements. In this case, the form of the matrix $\mathbf{A}_1$ will be as is shown in Fig. 6.

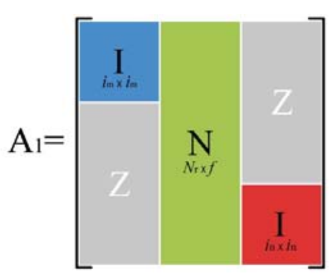


Fig. 6. Form of the matrix $\mathbf{A}_1$ for the structure shown in Fig. 4

Where **Z** is a zero matrix, f is the number of internal forces of the connecting elements of the two substructures.

Now for the formation of the matrix **N** we perform the following process:
If g is the set of DOFs for the connecting elements of substructures in local coordinate system and h is the set of DOFs of these elements in the global coordinate system (when all substructures are connected to each other), then a block of matrix **N** corresponding to the jth element can be formed as follows:

$$\mathbf{N}(g_j, h_j) = \mathbf{T}_j \tag{4}$$

$\mathbf{T}_j$ is the modified rotation matrix of the connecting jth element and can be formed as follows:

Consider $\mathbf{s}_j$ as the stiffness matrix of the jth element in a global coordinate system. One can re-arrange its rows and columns using a transformation matrix **R** such that the rank of $\mathbf{s}_1$ be the same as the dimension of this matrix and that of the $\mathbf{s}_j$.

$$\mathbf{s}_j^* = \mathbf{R}.\mathbf{s}_j.\mathbf{R}^t = \left[\begin{array}{c|c} \mathbf{s}_1 & \mathbf{s}_2 \\ \hline \mathbf{s}_2^t & \mathbf{s}_3 \end{array}\right] \tag{5}$$

The matrix $\mathbf{s}_j^*$ is obtained by applying the transformation **R** on the matrix $\mathbf{s}_j$. One can form the modified rotation matrix of the element in the following form:

$$\mathbf{T}_j = \mathbf{R}.\left[\mathbf{I} \mid \mathbf{s}_1^{-1}.\mathbf{s}_2\right]^t \tag{6}$$

In the above relationship, **I** is a identity matrix of dimension equal to that of $\mathbf{s}_1$.

b) Generalization of the method for joining more than two substructures

When some substructures are simultaneously connected to each other at some points, for the formation of the matrix **A** in a block form, the DOFs of the substructures should be classified as shown in Fig. 2. For each set of non-common nodes of a substructure (similar to the set of DOFs t_n in Fig. 2), one identity matrix with dimension of this DOFs should be assembled in matrix **A**. On the other hand, for each set of common node-set connecting two or more substructures (similar to DOFs i_n), there should be one matrix $\mathbf{A}_1$ in **A**. In the cases when more than two substructures are connected through a set of free nodes, then the matrix $\mathbf{A}_1$ will have the form $\mathbf{A}_1 = [\mathbf{I} \mid \mathbf{I} \mid \cdots \mid \mathbf{I}]$. This means the number of unit blocks in this matrix is increased by the number of added substructures. The dimension of each block is equal to the DOFs of these sets of common nodes in each substructure.

In some cases, the DOFs of the sets of common nodes in substructures are not equal to each other. In such cases, some columns of $\mathbf{A}_1$ corresponding to those DOFs which do not partcipate in connection should be deleted.

Therefore if a structure contains a non-common node sets corresponding to the substructures, and the substructures are connected to each other at b parts, then the form of equilibrium matrix will be as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_1 & & & & & \\ & (\mathbf{A}_1)_1 & & & & \\ & & \mathbf{I}_2 & & & \\ & & & \ddots & & \\ & & & & (\mathbf{A}_1)_b & \\ & & & & & \mathbf{I}_a \end{bmatrix} \tag{7}$$

The unit matrix in the equilibrium matrix corresponds to those DOFs of the substructures which are not connected to each other. As an example, consider the structure shown in Fig. 7 where for each free node of the shell element 6 DOFs are assumed.

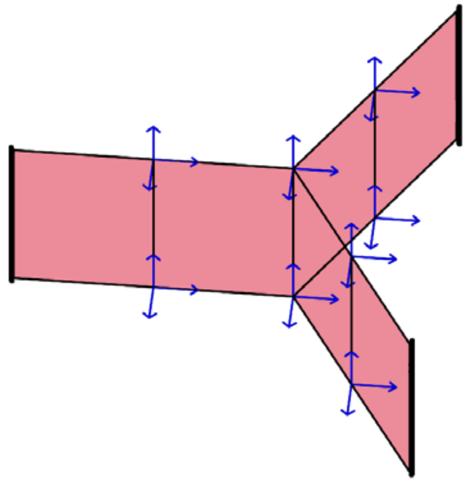


Fig. 7. A structure consisting of three connected substructures

However, if the substructures are considered separately, then writing the equilibrium for the DOFs at the middle part of the substructures, only the following unit matrix form will be obtained:

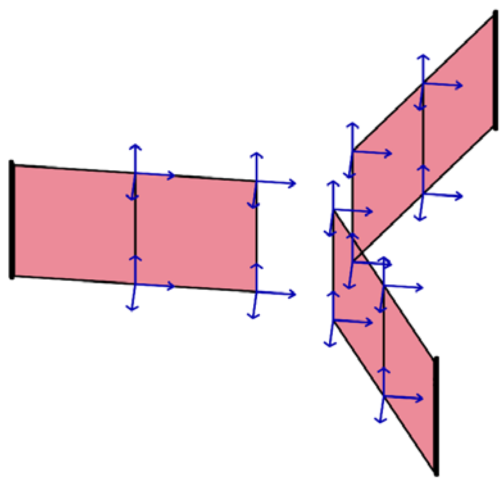


Fig. 8. The substructures forming the structure shown in Fig. 7 and the corresponding DOFs

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{I}_{36\times 36} & 0 \\ \hline 0 & \mathbf{A}_1 \end{array} \right] \quad ; \quad \mathbf{A}_1 = \left[\mathbf{I}_{12\times 12} \mid \mathbf{I}_{12\times 12} \mid \mathbf{I}_{12\times 12} \right]_{12\times 36} \tag{8}$$

It should be mentioned that in matrix A_1, the number of rows is the same as the DOFs of the common nodes of the substructures (when these are connected), and the number of columns is the same as the DOFs of the common nodes when the substructures are separate from each other.

3. OPERATORS REQUIRED FOR ANALYSIS

In this section, we describe the equilibrium matrix of a structure using the method presented in Section 2. First, the necessary operators are defined in the process of analysis. Then the way these operators are applied to the equilibrium matrix is described. Finally, a relationship is presented for calculating the nodal displacements of the entire structure using the concepts of the previous section.

The equations presented here were based on the SVD for the analysis of structures first developed by Pellegrino [21]. In this paper we want to simplify the equations for analysis of structures using the substructuring technique.

a) The SVD of the equilibrium matrix

In general, the equilibrium matrix of a structure can be decomposed into its singular values as follows:

$$\mathbf{A} = \mathbf{U}.\mathbf{W}.\mathbf{V}^{t} \tag{9}$$

Here, **U** and **V** are orthogonal matrices containing the left singular vectors and right singular vectors of the equilibrium matrix **A**. The matrix **W** is the matrix of the singular values of the equilibrium matrix. The matrix **V** can be partitioned into the following form based on the zero columns of **W**:

$$\mathbf{V} = \left[\mathbf{V}_{d} \mid \mathbf{V}_{z}\right] \quad ; \quad \mathbf{W} = \left[\mathbf{D} \mid \mathbf{Z}\right] \tag{10}$$

Where **D** is the square matrix of the non-zero singular values of the equilibrium matrix and **Z** is a zero matrix. As it was described, $\mathbf{V_d}$ and $\mathbf{V}_z$ are respectively the matrices containing the right singular vectors of **A** which correspond to the non-zero and zero singular values of the matrix **W**.

b) The pseudo-inverse of the equilibrium matrix

Using the SVD of the matrix **A**, the pseudo-inverse of this matrix can be obtained from the following equation:

$$\mathbf{Pinv}\ (\mathbf{A}) = \mathbf{V}_{d}.\mathbf{D}^{-1}.\mathbf{U}^{t} \tag{11}$$

It should be mentioned that due to the block form of the equilibrium matrix in Eq. (7), the SVD of this matrix needs only the decomposition of the $\mathbf{A}_1$. Therefore the SVD of $\mathbf{A}_1$, similar to Eq. (9), will be as follows:

$$\mathbf{A}_{1} = \mathbf{U}_{1}.\mathbf{W}_{1}.\mathbf{V}_{1}^{t} \tag{12}$$

Obviously the matrices $\mathbf{V}_1$, $\mathbf{W}_1$ and $\mathbf{U}_1$ have the same definitions as **V**, **W** and **U** for the matrix $\mathbf{A}_1$. Similarly we have:

$$\mathbf{V}_{1} = \left[\mathbf{V}_{1d} \mid \mathbf{V}_{1z}\right] \quad ; \quad \mathbf{W}_{1} = \left[\mathbf{D}_{1} \mid \mathbf{Z}\right] \quad ; \quad \mathbf{Pinv}(\mathbf{A}_{1}) = \mathbf{V}_{1d}.\mathbf{D}_{1}^{-1}.\mathbf{U}_{1}^{t} \tag{13}$$

c) Decomposition of the equilibrium matrix based on the decomposition of the matrix A_1.

Considering the block form of **A** in Eq. (2), the SVD of this matrix will also have a block form. Thus for this decomposition it is sufficient to decompose only the matrix $\mathbf{A}_1$. In subsequent step the constructed matrices are substituted in $\mathbf{A}_1$. The matrices $\mathbf{V}_d$, $\mathbf{V}_z$, **D**, and **Pinv**(**A**) for the structure of Fig. 1 have the following form:

$$\mathbf{V}_{d} = \begin{bmatrix} \mathbf{I}_{t_m \times t_m} & & \\ & \mathbf{V}_{1d} & \\ & & \mathbf{I}_{t_n \times t_n} \end{bmatrix} \quad ; \quad \mathbf{V}_{z} = \begin{bmatrix} \mathbf{Z} \\ \mathbf{V}_{1z} \\ \mathbf{Z} \end{bmatrix} \tag{14}$$

$$\mathbf{D}=\begin{bmatrix} \mathbf{I}_{t_m\times t_m} & & \\ & \mathbf{D}_1 & \\ & & \mathbf{I}_{t_n\times t_n} \end{bmatrix} \quad ; \quad \mathbf{Pinv(A)}=\begin{bmatrix} \mathbf{I}_{t_m\times t_m} & & \\ & \mathbf{Pinv(A_1)} & \\ & & \mathbf{I}_{t_n\times t_n} \end{bmatrix} \tag{15}$$

Since the dimension of the matrix $\mathbf{A}_1$ is much smaller compared to the dimension of the entire structure, the decomposition of this matrix is much simpler to perform. In this way, the formation of the decomposed form of the equilibrium matrix of the structure becomes simplified.

When a structure consists of more substructures, then the forms of the matrices in Eqs. (14 and 15) will expand. This means, for the formation of the above matrices instead of the matrix $\mathbf{A}_1$ in the block matrix $\mathbf{A}$, we put the matrices obtained by decomposition of the $\mathbf{A}_1$.

Only in the case of the matrix $\mathbf{V}_z$ should special attention be paid. In the process of the formation of this matrix one should eliminate all the rows and columns of $\mathbf{A}$ corresponding to the identity matrices and in place of the matrices $\mathbf{A}_1$, we should insert the corresponding matrices $\mathbf{V}_{1z}$.

4. NODAL DISPLACEMENT VECTOR OF THE STRUCTURE

In this section a method is presented for calculating the nodal displacements using the SVD of the equilibrium matrix. This method is taken from Ref. [21]. Using the matrices of the previous section, the nodal displacements $\boldsymbol{\Delta}$ of the structure can be expressed as:

$$\boldsymbol{\Delta} = \mathbf{Pinv(A)}^t.\mathbf{F}.[\mathbf{Pinv(A)}.\mathbf{P}-\mathbf{V}_z.(\mathbf{V}_z^{\,t}.\mathbf{F}.\mathbf{V}_z)^{-1}.(\mathbf{V}_z^{\,t}.\mathbf{F}.\mathbf{Pinv(A)}.\mathbf{P})] \tag{16}$$

This method is a combination of the stiffness and flexibility methods. The concept of using the equilibrium matrix comes from the flexibility method, while for formation of $\mathbf{F}$ the inverse of the stiffness matrix is utilized. Thus the stiffness method is used in the process of the flexibility approach. The proof of the above equations is provided in the Appendix A.

The dimension of the matrix $\mathbf{V}_z^{\,t}.\mathbf{F}.\mathbf{V}_z$ in the above relationship is equal to the sum of the DOFs of all the common nodes of the substructures. Here, $\mathbf{P}$ is the vector of external loads and $\mathbf{F}$ is the flexibility matrix of the substructures. As an example, for the structure shown in Fig. 1, if $\mathbf{F}_i$ is the flexibility of the ith substructure, we will have:

$$\mathbf{F}=\begin{bmatrix} \mathbf{F}_m & \\ & \mathbf{F}_n \end{bmatrix} \tag{17}$$

It is obvious that when some substructures are connected to each other simultaneously at some nodes, then the process of assembling the flexibility matrix $\mathbf{F}$ should be performed in the same order as that of the DOFs in the matrix $\mathbf{A}$.

The flexibility matrix of the substructures which are stable can be obtained from inverting the stiffness matrix of the substructure. For those substructures which are not supported in a stable manner, we will introduce a method of the next section. Using the modifying technique for the substructures which are not supported in a stable manner, sufficient condition will be produced. Another approach is due to the use of Felippa et al. [22] and Felippa and Park [23] for the formation of the flexibility matrices of such substructures.

It should be mentioned that in case we have some sets of connecting elements, the flexibility of these connecting elements should also be assembled in the matrix $\mathbf{F}$. As an example for the structure shown in Fig. 4, if F_{ei} is the flexibility matrix of the ith connecting element, then we will have:

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{\mathrm{m}} & & \\ & \mathbf{F}_{\mathrm{e}} & \\ & & \mathbf{F}_{\mathrm{n}} \end{bmatrix} \quad ; \quad \mathbf{F}_{\mathrm{e}} = \begin{bmatrix} \mathbf{F}_{\mathrm{e1}} & & & \\ & \mathbf{F}_{\mathrm{e2}} & & \\ & & \ddots & \\ & & & \mathbf{F}_{\mathrm{e}n} \end{bmatrix} \tag{18}$$

Therefore, for analysis of a structure we need the inverse of the stiffness matrices of the substructures and the SVD of the matrices $\mathbf{A}_1$ and the inverse of the matrix $\mathbf{V}_z^t . \mathbf{F} . \mathbf{V}_z$. Since the analysis of the entire structure is transformed to the analysis of substructures, the computational time will be reduced. At the same time, using the method of the next section, the inverse of the stiffness matrices of the substructures is performed with less computational storage.

5. MODIFICATION OF STRUCTURAL FORMS

For some structures the construction of canonical forms requires suitable transformation. There are many efficient methods for the analysis of regular structures. The structures described through group theory and graph products are some of such structures [16, 18-20].

Here, for modifying the form of the substructures the following two cases may arise:

In the first case, the structure contains some additional parts which can be separated from the main substructure in the form of new substructures. Therefore, the method of analysis will be exactly the same as that of Section 2.

In the second case, for transforming the structure to regular form, the addition of a set of substructures is required. In this case one can assume that two identical substructures are in the place of lack of elements, one with positive stiffness and another with negative stiffness. First, the structure is modified with the substructure having positive stiffness. The part with negative stiffness is considered as an independent substructure. Therefore, in the process of the analysis we will have to use two substructures. One is the modified (corrected) structure and the other one is the imaginary substructure with negative stiffness.

The first example of Section 6 illustrates the application of the method presented in this section for making the structure a regular one. Another application of this method is in the formation of the flexibility matrices of the substructures. In some cases the selected substructures are not stable and thus their inverse cannot be found to form their flexibility matrices. In this case, similar operations can be performed by addition of substructures which can provide suitable support conditions for the stability. This will be illustrated in the examples of the following section.

6. PRACTICAL EXAMPLES

In this section, some applications of the present method are presented in the form of practical examples. Then the results are compared for efficiency using the present method, a direct stiffness method written in Matlab, and ANSYS.

Example 1: The finite element model of a tunnel with semi-infinite space is shown in Fig. 9. As it can be seen, the opening of tunnel is at the middle of the model and model is discretized with a regular mesh. It is obvious that if we add a substructure in the form of the tunnel opening, it will become a regular model.

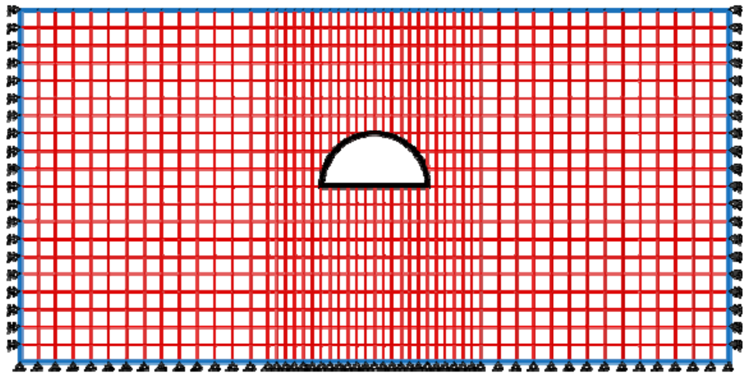


Fig. 9. Two dimensional view of the tunnel model

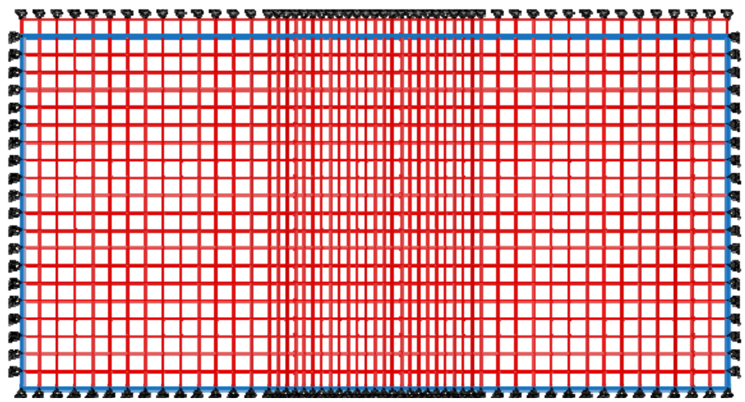


Fig. 10. The modified substructure

For this purpose, considering the method of Section 5, the set of substructures are considered for modifying the structural form, as shown in Fig. 11.

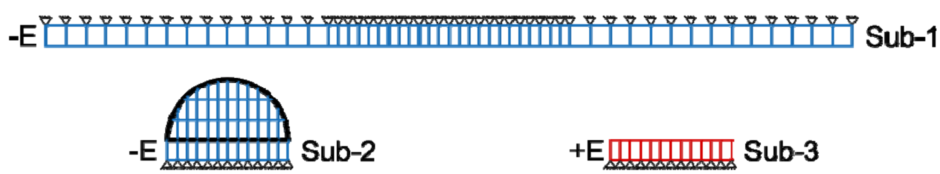


Fig. 11. Modifying substructures of the tunnel model

In Fig. 10 the entrance of the tunnel is filled with a set of elements. Instead, in Fig. 11 an equivalent substructure with negative stiffness is defined and called Sub-2. This substructure does not have sufficient supporting condition for geometric stability. Therefore a row of elements with negative stiffness is added with appropriate supporting condition. The Sub-3 substructure with positive stiffness is considered for nullifying the effect of the imaginary elements in order to make the Sub-2 stable. The substructure Sub-1 is used for fixing the upper nodes of the model to provide the symmetry property. The expression E shown in Fig. 11 shows the stiffness of the element. In fact –E means negative stiffness for the element.

In what follows, the matrices $\mathbf{A}_1$ are formed for the constituting substructures and using their SVD, the structural analysis is performed using the present method. In case of having displacement boundary condition in the FEM, one can transform the corresponding displacements with equivalent external forces applied at the DOFs of the structure. Based on this, if $\mathbf{S}$ is the overall stiffness matrix and the indices f and s are the DOFs of free and fixed nodes of the structure, then the equivalent external load and the displacement boundary condition, $\mathbf{P}_e$, will be in the following form:

$$\mathbf{S}=\left[\begin{array}{c|c}\mathbf{S}_{ff} & \mathbf{S}_{fs}\\ \hline \mathbf{S}_{sf} & \mathbf{S}_{ss}\end{array}\right] \quad ; \quad \mathbf{P}_{\mathrm{e}}=\mathbf{P}-\mathbf{S}_{fs}\,.\boldsymbol{\Delta}_s \tag{19}$$

Where **P** is the external load vector applied to the structure, and Δ_s is the vector of displacements of the supports. Therefore, it is sufficient to use $\mathbf{P}_{\mathrm{e}}$ in place of the external load vector in this method.

It is obvious that the inverse of the stiffness matrices of the Subs-1,2 and 3 can easily be calculated, because of having small dimensions. For the formation of the stiffness matrix of the modified substructure shown in Fig. 10, one can use the method of Ref. [16] which is based on some concepts of the graph products. For this purpose, the stiffness matrix **S** of this substructure is expressed in the following form:

$$\mathbf{S}_{mn}=\begin{bmatrix}\mathbf{A}_m & \mathbf{B}_m & & & \mathbf{0}\\ \mathbf{B}_m & \mathbf{A}_m & \mathbf{B}_m & & \\ & \mathbf{B}_m & . & & \\ & & & . & \mathbf{B}_m\\ \mathbf{0} & & & \mathbf{B}_m & \mathbf{A}_m\end{bmatrix}_n=\mathbf{I}_n\otimes\mathbf{A}_m+\mathbf{T}_n\otimes\mathbf{B}_m \tag{20}$$

$$\mathbf{S}_{mn}=\mathbf{F}_n(\mathbf{A}_m,\mathbf{B}_m,\mathbf{A}_m) \quad ; \quad \mathbf{I}_n=\mathbf{F}_n(1,0,1) \quad ; \quad \mathbf{T}_n=\mathbf{F}_n(0,1,0)$$

It should be mentioned that the $\mathbf{F}_n$ is defined as the following block matrix form:

$$\mathbf{F}_n(\mathbf{A}_m,\mathbf{B}_m,\mathbf{C}_m)=\begin{bmatrix}\mathbf{A}_m & \mathbf{B}_m & & & \\ \mathbf{B}_m & \mathbf{C}_m & \mathbf{B}_m & & \\ & \mathbf{B}_m & \ddots & \ddots & \\ & & \ddots & \mathbf{C}_m & \mathbf{B}_m\\ & & & \mathbf{B}_m & \mathbf{A}_m\end{bmatrix} \tag{21}$$

The blocks $\mathbf{A}_m$ and $\mathbf{B}_m$ have dimension m and matrix **S** is a tri-diagonal matrix. $\mathbf{I}_n$ and $\mathbf{T}_n$ are blocks of dimension n. In fact, **S** is formed of n blocks of $m\times m$. The eigenvalues of this matrix can be obtained from the following relationships:

$$\lambda_{\mathbf{S}}=\bigcup_{i=1}^{n}\lambda_{\mathbf{S}_i} \quad ; \quad \mathbf{S}_i=\mathbf{A}_m+\lambda_i(\mathbf{T}_n).\mathbf{B}_m \quad ; \quad \lambda_i(\mathbf{T}_n)=2\cos\frac{i\pi}{n+1} \tag{22}$$

Consider **v** as the eigenvector of $\mathbf{S}_i$ and **u** as the eigenvector of the matrix $\mathbf{T}_n$. Then $\boldsymbol{\varphi}=\mathbf{u}\otimes\mathbf{v}$ will be the eigenvector of the matrix **S**.

Obviously if $\boldsymbol{\varphi}$ is the matrix of the eigenvectors and $\boldsymbol{\lambda}$ is a diagonal matrix with its entries being the corresponding eigenvalues, then we will have $\mathbf{S}=\boldsymbol{\varphi}.\boldsymbol{\lambda}.\boldsymbol{\varphi}^{\mathrm{t}}$. Since the eigenvalues of $\mathbf{S}^{-1}$ are the inverse of the eigenvalue of **S** and their eigenvectors are identical, therefore:

$$\mathbf{S}^{-1}=\boldsymbol{\varphi}.\boldsymbol{\lambda}^{-1}.\boldsymbol{\varphi}^{\mathrm{t}}=\boldsymbol{\varphi}.\begin{bmatrix}1/\lambda_1 & & & & 0\\ & 1/\lambda_2 & & & \\ & & . & & \\ & & & . & \\ 0 & & & & 1/\lambda_{m\times n}\end{bmatrix}.\boldsymbol{\varphi}^{\mathrm{t}} \tag{23}$$

Where $\boldsymbol{\lambda}^{-1}$ can be obtained by finding the inverse of the diagonal entries $\boldsymbol{\lambda}$. Therefore the inverse of the stiffness matrix $\mathbf{S}^{-1}$ for the substructure shown in Fig. 10 can be obtained using n eigenvalues of an $m\times m$ matrix.

In Fig. 12, the computational time is compared for the analysis of problems with different DOFs using the present method and the direct method. In this Figure, DOF is the degree of freedom of the model, “Direct M.” refers to the direct method, and “Present M.” refers to the method presented in this paper.

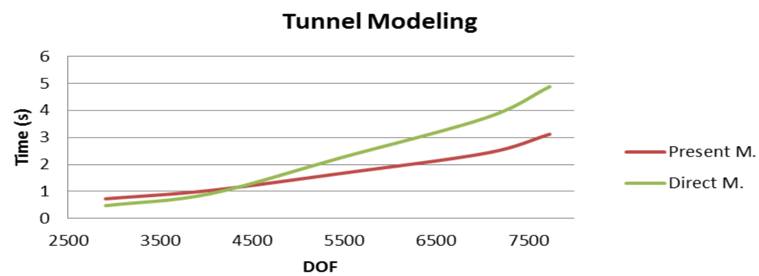


Fig. 12. Comparison of the computational time for the analysis of the tunnel, using the present method and a direct method (UMFPACK)

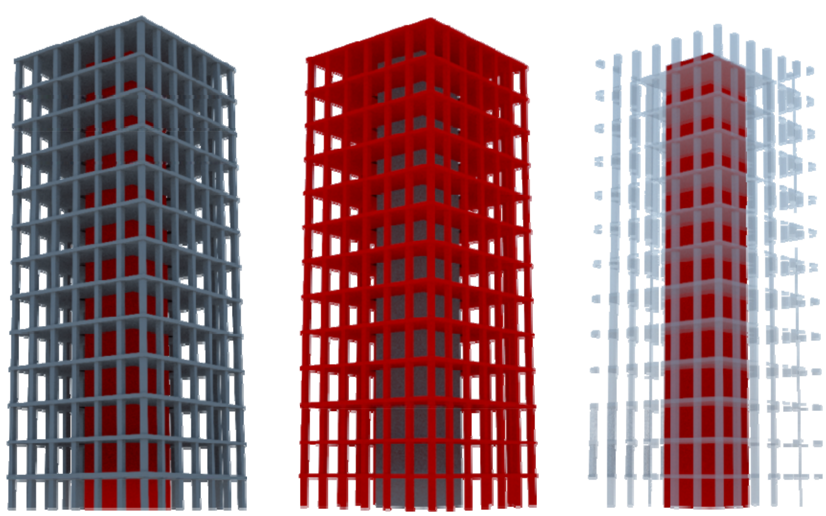
For direct analysis of the model, the UMFPACK is utilized. UMFPACK is an Unsymmetric Multi Frontal direct solver for sparse matrices developed by Davis [26]. This package consists of a set of routines for solving systems of linear equations. It can be observed that as the number of DOFs of the structure increases the difference between the computational time for the present method and the UMFPACK increases.

In fact, the rate of the change of the computational time for the conventional method is increasing, while for the present approach this rate is nearly constant.

In this example, the opening need not be a half-circle. Obviously the presence of a few irregular elements in this opening can easily deal with the concept of connecting elements (Figs. 4 and 5). Thus the form of the opening is not an important issue in the utilization of the presented method.

Example 2: One of the applications of the present method is in efficient analysis of space frame having a core containing shear walls.

For this structure, since the dimensions of the structure are high, using the present method is of great importance. Here the shear wall has higher share from the total DOFs, which can be separately analyzed. Figure 13 shows a 3D tall building and the corresponding substructures. Here, we have a 13 storey frame with 4 shear walls. The walls form a box type of structure with rectangular cross section and its thickness changes in each story.



(a) 3D view of a frame structure (b) Bending frame substructure (c) Shear wall substructure

Fig. 13. Three dimensional view of a space frame and its constituting substructures

Using a regular meshing and modifying the support conditions at the top of the shear wall, the form of this substructure becomes canonical and it can easily be analyzed by methods of Ref. [19]. For a better illustration of the method used for modification, a 2D form of this tower is shown in Figs. 14 and 15.

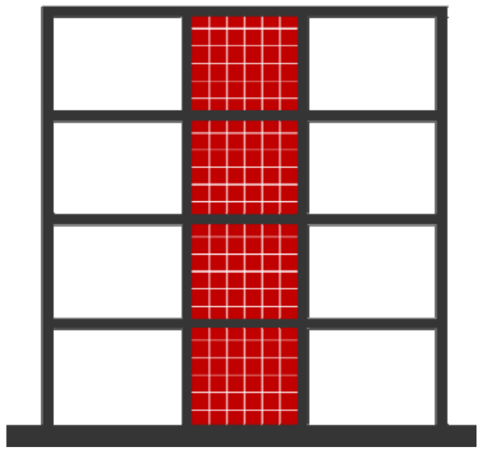
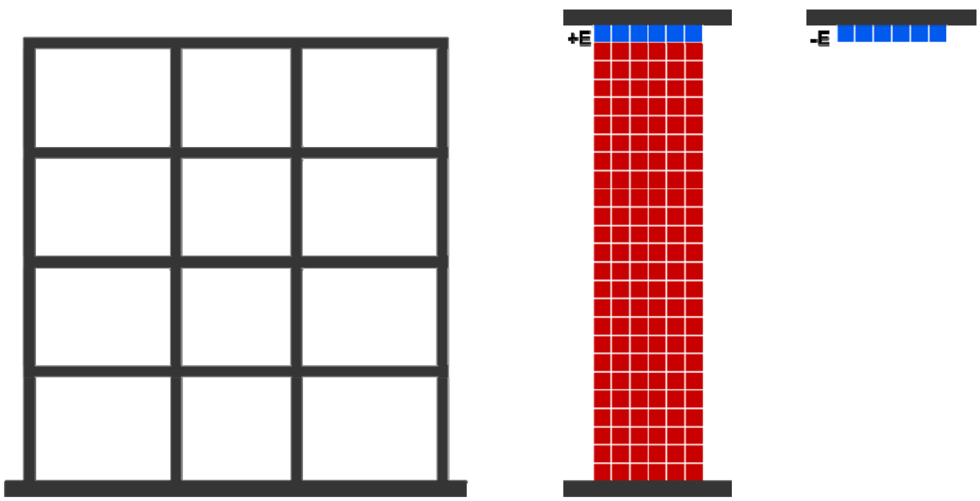


Fig. 14. Two dimensional view of a frame with a shear wall



(a) Frame substructure

(b) Shear wall substructure

(c) Modifying part for the shear wall substructure

Fig. 15. Two dimensional view of the substructures of the structure

Using the method of Section 5, a row of elements with suitable support condition is added to the top part of the shear wall. Then an identical substructure with negative stiffness is added as shown in Fig. 15. In the following we construct the equilibrium matrix of these substructures.

With a suitable assumption one can say that the total DOFs of the entire structure is 41262 from which the share of the bending frame is 3822. The two substructures, frame and the shear wall, are connected to each other through 624 DOFs.

Considering the form of the shear walls, it can be decomposed into 8 identical substructures. Form of each of these substructures can be modified as illustrated in Fig. 15 using an additional modifying substructure. Here the equilibrium matrix is composed of 16 submatrices $\mathbf{A}_1$, from which only two are non-identical. One is for connecting the shear walls and frame to each other and the other is for connecting the modifying substructures to the shear walls.

For the formation of the inverse of the stiffness matrix of the shear wall in Fig. 15, we utilize the method of Ref. [19] in the following form: The stiffness matrix of each shear wall is as $\mathbf{S}_{mn} = \mathbf{F}_n(\mathbf{A}_m, \mathbf{B}_m, \mathbf{C}_m)$. This matrix has n blocks of dimension $m \times m$ in each row and column. First we assume n to be even, i.e. $n=2k$. As an example, for $n=4$ this form will be as follows:

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{C} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{A} \end{bmatrix} \tag{24}$$

Exchanging the 3rd and 4th rows and columns in block form, the eigenvalues of the matrix remain unaltered. Then **S** will have the following form:

$$\mathbf{S} = \left[\begin{array}{cc|cc} \mathbf{A} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{0} & \mathbf{B} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{B} & \mathbf{B} & \mathbf{C} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{N} \\ \hline \mathbf{N} & \mathbf{M} \end{array}\right] = \mathbf{I}_2 \otimes \mathbf{M} + \mathbf{T}_2 \otimes \mathbf{N} \tag{25}$$

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \mathbf{T}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{26}$$

Using λ to represent the eigenvalues, similar to Eq. (22) one can show that:

$$\lambda_{\mathbf{S}} = \bigcup_{i=1}^{2} \lambda_{\mathbf{S}_i} \quad ; \quad \mathbf{S}_i = \mathbf{M} + \lambda_i(\mathbf{T}_2).\mathbf{N} \quad ; \quad \lambda_i(\mathbf{T}_2) = \{1, -1\} \tag{27}$$

The eigenvalues of the matrix **S** and the blocks $\mathbf{S}_i$ are the same. In fact the dimensions of the required matrices reduce to half. On the other hand, if **v** is the eigenvalue of $\mathbf{S}_i$ and **u** is that of $\mathbf{T}_2$, then $\boldsymbol{\varphi} = \mathbf{u} \otimes \mathbf{v}$ is the eigenvector of the matrix **S**. In general case, when $n=2k$ it is enough to exchange the $(k+1)$th row and column with those of $2k$th to arrive at a Form II canonical form. It can be shown that in this case **M** and **N** will be as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & & & & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{B} & & & \\ & \mathbf{B} & . & . & & \\ & & . & . & . & \\ & & & . & . & \mathbf{B} \\ \mathbf{0} & & & & \mathbf{B} & \mathbf{C} \end{bmatrix}_{k\ \text{blocks}} , \quad \mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & & & & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & & \\ & \mathbf{0} & . & . & & \\ & & . & . & . & \\ & & & . & . & \mathbf{0} \\ \mathbf{0} & & & & \mathbf{0} & \mathbf{B} \end{bmatrix}_{k\ \text{blocks}} \tag{28}$$

Matrix **N** contains mainly zero blocks except in the block corresponding to the row and column k which is the same as **B**. This form is the same as the canonical form II of Ref. [19].

Now we assume n to be odd, i.e $n=2k-1$. As an example, for $n=3$ we will have

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{B} \\ \mathbf{0} & \mathbf{B} & \mathbf{A} \end{bmatrix} \tag{29}$$

Here, first a row and column of zero blocks are added between the 2nd and 3rd blocks, and repeat the previous process. Since the number of rows and columns are even, with some row and column operations and exchanging the 3rd and 4th rows and columns, the following matrix can be constructed:

$$\mathbf{S} = \left[\begin{array}{cc|cc} \mathbf{A} & \mathbf{B} & \mathbf{0} & \mathbf{B} \\ \frac{\mathbf{B}}{2} & \frac{\mathbf{C}}{2} & \frac{\mathbf{B}}{2} & \frac{\mathbf{C}}{2} \\ \hline \mathbf{0} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \frac{\mathbf{B}}{2} & \frac{\mathbf{C}}{2} & \frac{\mathbf{B}}{2} & \frac{\mathbf{C}}{2} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{N} \\ \hline \mathbf{N} & \mathbf{M} \end{array}\right] \equiv \left[\begin{array}{c|c} \mathbf{M}+\mathbf{N} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{M}-\mathbf{N} \end{array}\right] \tag{30}$$

Here the sign $\equiv$ is use for equivalence of two matrices.

In general case for $n = 2k-1$, repeating a process similar to the previous case, $\mathbf{M}$ and $\mathbf{N}$ can be obtained as:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & & & & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{B} & & & \\ & \mathbf{B} & . & . & & \\ & & . & . & \mathbf{B} & \\ & & & \mathbf{B} & \mathbf{C} & \mathbf{B} \\ \mathbf{0} & & & & \frac{\mathbf{B}}{2} & \frac{\mathbf{C}}{2} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & & & & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & & \\ & \mathbf{0} & . & . & & \\ & & . & . & . & \mathbf{0} \\ & & & . & . & \mathbf{B} \\ \mathbf{0} & & & \mathbf{0} & \frac{\mathbf{B}}{2} & \frac{\mathbf{C}}{2} \end{bmatrix} \tag{31}$$

Where **N** has only three non-zero blocks.

Each of these matrices consists of *k* blocks. Obviously after calculating the eigenvalues of **M**+**N** and **M**–**N**, we will have *m* additional zero eigenvalues created because of the addition of zero block rows and columns. Having the eigenvalues and eigenvectors of **S**, the inverse of the stiffness matrix of the shear walls will be obtained using Eq. (23).

In relation to the operational cost, since the stiffness matrices of the substructure of the shear walls are identical only the eigenvalues of two 2844×2844 blocks for the inversion of their stiffness matrices need to be calculated. For the frame substructure, the inversion of a matrix of dimension 3822×3822 is needed.

As we mentioned before, here the connection of substructures has two forms. First the connection of the shear wall together and to the frame will have an $\mathbf{A}_1$ matrix of dimension 468×1014, and second the connection of the correcting substructure of the walls as shown in Fig. 15. The matrix $\mathbf{A}_1$ for this connection has dimension 66×132. The SVD of these two matrices is constructed. Since we have 8 shear walls in the structure, 8 pairs of the matrix $\mathbf{A}_1$ will exist in the matrix **A**.

Finally the inverse of the matrix $\mathbf{V}_z^t\,.\mathbf{F}.\mathbf{V}_z$ will have dimension 4912 ×4912. While for direct analysis method we have to solve a set of equations with a matrix of dimension 41362×41362. If the process is repeated for different DOFs, the variation of the computational time will be as shown in Fig. 16.

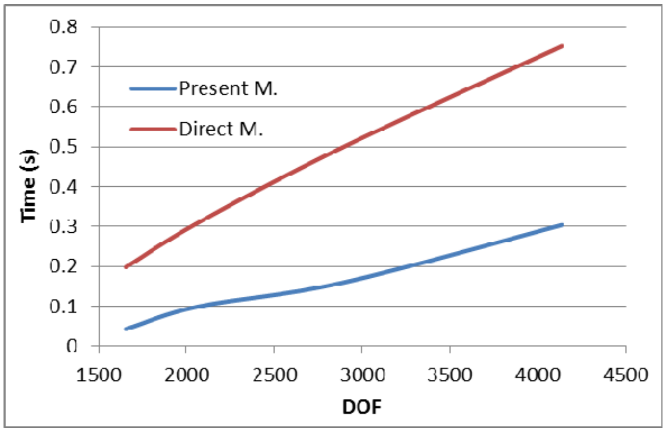


Fig. 16. Comparative study of the present method and direct approach for different DOFs

Example 3: A base plate is considered as shown in Fig. 17. For the analysis of this model, different parts are considered as substructures. For filling the holes, the method of Section 5 is used. In this way the analysis is transformed into the analysis of some substructures.

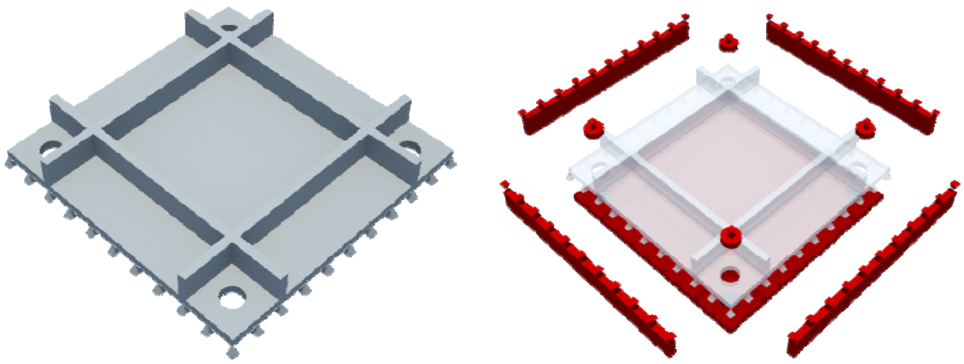


Fig. 17. A three dimensional view of a base plate and its constituting substructures

Here, the substructures are divided into two groups. The first group contains the main parts of the base plate, and the second group consists of those substructures which provide support system for the substructures of the first group.

As can be seen from Fig. 17, some of the substructures are identical. Therefore the calculation of one flexibility matrix is sufficient. We have 4 independent flexibility matrices, two of which have very low dimensions. One of these corresponds to the holes substructures and the other belongs to the support conditions substructure of the vertical plates. The bigger flexibility matrices correspond to horizontal and vertical plates which can be calculated with a regular meshing, similar to Example 1 of this section.

The computational time for the analysis of structures with different numbers of DOFs for following cases is shown in Fig. 18:

1. Simultaneous use of SVD, and graph product methods (present Method).
2. Substructuring technique of Ref. [24].
3. Condensation method of Ref. [25]
4. UMFPACK method of Ref. [26]

An efficient ordering is performed in direct analysis of LU decomposition by UMFPACK. As mentioned before the UMFPACK is an Unsymmetric Multifrontal sparse LU factorization package [26].

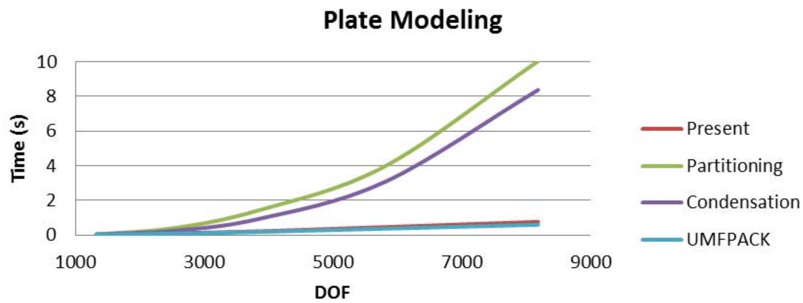


Fig. 18. Comparison of the computational time for the analysis of Example 3, using different methods

For this problem the present methods and the UMFPACK perform nearly the same, however, these methods perform much better than the partitioning and condensation approaches.

7. CONCLUDING REMARKS

In this paper, an efficient method is developed for separation, and independent analysis of substructures. Here, the analysis of the structure is performed by formation of the equilibrium matrix of the substructures and performing its SVD. Considering the block diagonal form, it is only necessary to decompose small parts of this matrix. Inverting the stiffness matrices of the substructure, the analysis of the entire structure is performed.

As shown in the practical examples of Section 6, saving a large amount of structural data is not necessary by using this method. Hence, the analysis costs decreases significantly. Also, by easy separation of the sub-structures in this method, one can find the inverse of their stiffness matrices through the previously developed fast methods [19]. Furthermore, this method can easily be added to structural analyzer software, since the applied matrices can systemically be formed.

Finally, the major difference between this method and the others belongs to the ability of the structural modification. As shown, by this method one can modify the topology of a structure by adding some parts in a way that the use of fast analysis methods becomes feasible.

Acknowledgements: The first author would like to acknowledge the financial support of the University of Tehran for this research. The second author is grateful to the Iran National Science Foundation for the support.

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APPENDIX A: Proof of Eq. (16)

The proof presented in this appendix is based on the equations developed in [27]. To find Eq. (16) of the present paper, the following steps should be performed:

Step 1: Eq. (16) and Eq. (12) of ref. [27] are as follows:

$$Q = pinv(A).P + V_z.\alpha \tag{A1}$$

$$\alpha = -(V_z^t.F.V_z)^{-1}.(V_z^t.F.pinv(A).P) \tag{A2}$$

Substituting Eq. (A2) in Eq. (A1) we obtain:

$$Q = pinv(A).P - V_z.(V_z^t.F.V_z)^{-1}.V_z^t.F.pinv(A).P \tag{A3}$$

Step 2: Consider the Eq. (14) of ref. [27] as

$$\delta = F.Q \tag{A4}$$

Substituting (A3) in (A4) we have

$$\delta = F.pinv(A).P - F.V_z.(V_z^t.F.V_z)^{-1}.V_z^t.F.pinv(A).P \tag{A5}$$

Step 3: Consider the Eq. (19) of ref. [27] as

$$\Delta = Pinv\ (A^t).\delta \tag{A6}$$

Substituting (A5) in (A6), we obtain Eq. (16) of the present paper as:

$$\Delta = Pinv\ (A^t).F.(pinv(A).P - V_z.(V_z^t.F.V_z)^{-1}.V_z^t.F.pinv(A).P) \tag{A7}$$