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# THREE-DIMENSIONAL ELASTICITY SOLUTION OF TRANSVERSELY ISOTROPIC RECTANGULAR PLATES WITH VARIABLE THICKNESS*

Y. P. XU[1] AND D. ZHOU[2**]

[1]School of Science, Nanjing University of Science and Technology,
210094 Nanjing, People's Republic of China
[2]College of Civil Engineering, Nanjing University of Technology,
210009 Nanjing, People's Republic of China
Email: dingzhou57@yahoo.com

**Abstract -** This paper studies the stress and displacement distributions of transversely isotropic rectangular plates with continuously varying thickness and simply supported at four edges. On the basis of three-dimensional elasticity theory, the general expressions for the displacements and stresses of the plate under static loads, which exactly satisfy the governing differential equations and the edge conditions of the plate, are analytically derived. The unknown coefficients in the solutions are approximately determined by using the double Fourier sinusoidal series expansions to the upper surface and lower surface conditions of the plate. Convergence and comparison studies demonstrate the correctness and effectiveness of the proposed method.

**Keywords–** Rectangular plate, transversely isotropic material, variable thickness, Fourier expansion, three-dimensional elasticity solution

## 1. INTRODUCTION

Plates are the most commonly used structural components in aerospace, mechanical and civil engineering. In most cases, the plates have to carry various loads, therefore, a thorough understanding of their mechanics characteristics is essential for designers. Although plates with constant thickness have been widely used, the variable thickness plates have also received a lot of attention from designers and researches. The use of variable thickness can help the designer to reduce the weight of the structure. For cases where reduction of weight is of high importance, such as space structures, this type of plate is the best choice. For the elastic analysis of plates with variable thickness, it is very difficult to obtain the exact solutions, even using the classical plate theory. Conway [1-2] studied the elastic bending of axisymmetric circular plates with tapered thickness. Ohga and Shigematsu [3] used a combination of boundary element and transfer matrix method to solve rectangular plates with variable thickness, however, this method provided a solution only for a special case. Zenkour [4] presented an exact solution for the bending of thin rectangular plates with linear and quadratic thickness variations. Eisenberger and Alexandrov [5] presented accurate solutions for bifurcation buckling loads of rectangular thin plates with thickness that varies in the directions parallel to the two sides of the plate. Ghorashi and Daneshpazhooh [6] carried out the limit analysis for variable thickness circular plates. Efraim and Eisenberger [7] studied the vibration of thick annular plates with variable thickness, respectively, made of isotropic material and FGM. Sakiyama and Huang [8] proposed an approximate method for analyzing the free vibration of thin and moderately thick rectangular plates with arbitrary variable thickness. Gagnon et al. [9] used the finite strip element to

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**Corresponding author

analyze thick rectangular plates with variable thickness. Liew et al. [10] presented a semi-analytical solution to analyze the vibration of rectangular plates with abrupt thickness variation. Liu and Liew [11] studied the free vibration of discontinuous Mindlin plates by differential quadrature element method. Liew et al. [12] studied the free vibration of stress-free hollow cylinders with arbitrary cross section. Moreover, they studied the transverse vibration of trapezoidal plates of variable thickness [13-14]. Hatami et al. [15] studied stability and vibration of elastically supported, axially moving orthotropic plates. Yasarkaltakci and Arslan [16] studied stress concentrations of symmetrically laminated composite plates containing circular holes. Recently, Xu and Zhou [17] presented the three-dimensionally elasticity solution for simply supported isotropic rectangular plates with variable thickness.

In general, the structure and problem of analysis can be solved using a standard FE package. However, the common FE package may not be easy to use when dealing with functionally graded materials. Moreover, the analytical method is more convenient for making the parameterized study and can provide more accurate results than the FE method. In the present study, the general expressions for the displacements and stresses, which exactly satisfy the governing differential equations and the simply supported conditions at four edges of the plate, have been analytically derived. The unknown coefficients in the solutions are approximately determined by using the expansions of double Fourier sinusoidal series to the upper surface and lower surface conditions of the plate.

## 2. ELASTICITY SOLUTIONS

Consider a continuously varying thickness rectangular plate with length *a*, width *b* and variable thickness, as shown in Fig. 1. The plate is simply supported at four edges and has the thickness *H* at one side. The upper and lower surfaces of the plate are described by the continuous functions $f_1(x, y)$ and $f_2(x, y)$ respectively. The upper surface of the plate is subjected to the transverse load $q(x, y)$ in the z direction.

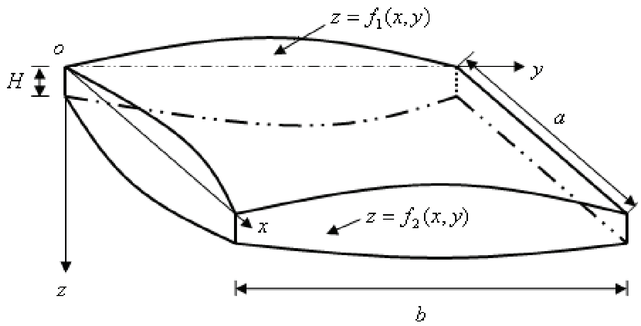



Fig. 1. Transversely isotropic rectangular plate with continuously varying thickness

In the Cartesian coordinate system, the three-dimensional constitutive relations of a transversely isotropic elastic body are given as follows:

$$\sigma_x = C_{11}\frac{\partial u}{\partial x} + C_{12}\frac{\partial v}{\partial y} + C_{13}\frac{\partial w}{\partial z}, \ \tau_{xy} = \frac{C_{11} - C_{12}}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$
$$\sigma_y = C_{11}\frac{\partial v}{\partial y} + C_{12}\frac{\partial u}{\partial x} + C_{13}\frac{\partial w}{\partial z}, \ \tau_{yz} = C_{44}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \tag{1}$$
$$\sigma_z = C_{33}\frac{\partial w}{\partial z} + C_{13}\frac{\partial u}{\partial x} + C_{13}\frac{\partial v}{\partial y}, \ \tau_{xz} = C_{44}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})$$

In the above equation, $\sigma_x$, $\sigma_y$ and $\sigma_z$ represent the stresses in the x-, y- and z- directions and $\tau_{xy}$, $\tau_{xz}$, $\tau_{yz}$ are the shear stresses. $C_{11}$, $C_{12}$, $C_{13}$, $C_{33}$ and $C_{44}$ are the elastic constants to describe the mechanical properties of transversely isotropic material. The displacements in the x-, y-, and z- directions are represented as $u$, $v$, and $w$, respectively.

For an elastic body in the absence of body forces, the differential equations of equilibrium can be written in terms of displacements as follows

$$\begin{aligned}
&C_{11}\frac{\partial^2 u}{\partial x^2}+\frac{C_{11}-C_{12}}{2}\frac{\partial^2 u}{\partial y^2}+C_{44}\frac{\partial^2 u}{\partial z^2}+\frac{C_{11}+C_{12}}{2}\frac{\partial^2 v}{\partial x\partial y}+(C_{13}+C_{44})\frac{\partial^2 w}{\partial x\partial z}=0\\
&C_{11}\frac{\partial^2 v}{\partial y^2}+\frac{C_{11}-C_{12}}{2}\frac{\partial^2 v}{\partial x^2}+C_{44}\frac{\partial^2 v}{\partial z^2}+\frac{C_{11}+C_{12}}{2}\frac{\partial^2 u}{\partial x\partial y}+(C_{13}+C_{44})\frac{\partial^2 w}{\partial y\partial z}=0\\
&C_{33}\frac{\partial^2 w}{\partial z^2}+C_{44}\frac{\partial^2 w}{\partial x^2}+C_{44}\frac{\partial^2 w}{\partial y^2}+(C_{13}+C_{44})\frac{\partial^2 u}{\partial x\partial z}+(C_{13}+C_{44})\frac{\partial^2 v}{\partial y\partial z}=0
\end{aligned}\tag{2}$$

When the plate is simply supported at four edges, its edge conditions are simulated by

$$\begin{aligned}
&\sigma_x=0,\ v=w=0\ \text{ at } x=0,\ a\\
&\sigma_y=0,\ u=w=0\ \text{ at } y=0,\ b
\end{aligned}\tag{3}$$

where $a$ and $b$ are, respectively, the length and width of the plate.

Using the method of separation of variables and considering the simply supported conditions at four ends of the plate, the displacement distributions have the following form:

$$\begin{aligned}
u&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
v&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\\
w&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}
\end{aligned}\tag{4}$$

where $U_{mn}(z), V_{mn}(z)$ and $W_{mn}(z)$ are the unknown functions about the coordinate $z$. It can be seen that Eq. (4) exactly satisfies Eq. (3). Letting $a_{mn}=\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}}$ and substituting Eq. (4) into Eq. (2), one has

$$\begin{aligned}
&U''_{mn}(z)-\frac{C_{11}-C_{12}}{2C_{44}}a_{mn}{}^2U_{mn}(z)-\frac{m^2\pi^2}{a^2}\frac{C_{11}+C_{12}}{2C_{44}}U_{mn}(z)-\\
&\qquad\frac{mn\pi^2}{ab}\frac{C_{11}+C_{12}}{2C_{44}}V_{mn}(z)+\frac{m\pi}{a}\frac{C_{13}+C_{44}}{C_{44}}W'_{mn}(z)=0\\
&V''_{mn}(z)-\frac{C_{11}-C_{12}}{2C_{44}}a_{mn}{}^2V_{mn}(z)-\frac{n^2\pi^2}{b^2}\frac{C_{11}+C_{12}}{2C_{44}}V_{mn}(z)-\\
&\qquad\frac{mn\pi^2}{ab}\frac{C_{11}+C_{12}}{2C_{44}}U_{mn}(z)+\frac{n\pi}{b}\frac{C_{13}+C_{44}}{C_{44}}W'_{mn}(z)=0\\
&W''_{mn}(z)-\frac{C_{44}}{C_{33}}a_{mn}{}^2W_{mn}(z)-\frac{C_{13}+C_{44}}{C_{44}}\frac{m\pi}{a}U'_{mn}(z)-\frac{C_{13}+C_{44}}{C_{44}}\frac{n\pi}{b}V'_{mn}(z)=0
\end{aligned}\tag{5}$$

Simultaneously solving Eq. (5) and eliminating functions $U_{mn}(z)$ and $V_{mn}(z)$, one has

$$W_{mn}''''(z)-t_1 a_{mn}^2 W_{mn}''(z)+t_2 a_{mn}^4 W_{mn}(z)=0 \tag{6}$$

$$t_1=\frac{C_{44}^2-(C_{13}+C_{44})^2+C_{11}C_{33}}{C_{33}C_{44}}, \quad t_2=\frac{C_{11}}{C_{33}} \tag{7}$$

The solution form of Eq. (6) is dependent on the sign of the coefficient $t_1^2-4t_2$. Therefore, there are three possible solutions:

*Solution I*: when $t_1^2-4t_2>0$

$$W_{mn}(z)=e^{r_1 a_{mn} z}A_{mn}+e^{-r_1 a_{mn} z}B_{mn}+e^{r_2 a_{mn} z}C_{mn}+e^{-r_2 a_{mn} z}D_{mn} \tag{8}$$

in which,

$$r_1=\sqrt{\frac{t_1+\sqrt{t_1^2-4t_2}}{2}}, \quad r_2=\sqrt{\frac{t_1-\sqrt{t_1^2-4t_2}}{2}} \tag{9}$$

*Solution II*: when $t_1^2-4t_2=0$

$$W_{mn}(z)=e^{r a_{mn} z}A_{mn}+e^{-r a_{mn} z}B_{mn}+z e^{r a_{mn} z}C_{mn}+z e^{-r a_{mn} z}D_{mn} \tag{10}$$

in which,

$$r=\sqrt{t_1/2} \tag{11}$$

*Solution III*: when $t_1^2-4t_2<0$

$$\begin{aligned}W_{mn}(z)=&\cos(r_2 a_{mn} z)e^{r_1 a_{mn} z}A_{mn}+\sin(r_2 a_{mn} z)e^{r_1 a_{mn} z}B_{mn}+\\&\cos(r_2 a_{mn} z)e^{-r_1 a_{mn} z}C_{mn}+\sin(r_2 a_{mn} z)e^{-r_1 a_{mn} z}D_{mn}\end{aligned} \tag{12}$$

in which

$$r_1=\frac{\sqrt{t_1+2\sqrt{t_2}}}{2}, \quad r_2=\frac{\sqrt{-t_1+2\sqrt{t_2}}}{2} \tag{13}$$

Substituting Eqs. (8), (10) and (12) into Eq. (5) respectively, the solutions of $U_{mn}(z)$ and $V_{mn}(z)$ can be obtained for each case. Taking the case $t_1^2-4t_2>0$ as an example, we have

$$\begin{aligned}u(x,y,z)=&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[\\&\frac{m\pi}{a}\frac{R_1}{\alpha_{mn}}(e^{r_1 a_{mn} z}A_{mn}-e^{-r_1 a_{mn} z}B_{mn})+\frac{m\pi}{a}\frac{R_2}{\alpha_{mn}}(e^{r_2 a_{mn} z}C_{mn}-e^{-r_2 a_{mn} z}D_{mn})+\\&e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn} z}E_{mn}+e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn} z}F_{mn}]\\v(x,y,z)=&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}[\\&\frac{n\pi}{b}\frac{R_1}{\alpha_{mn}}(e^{r_1 a_{mn} z}A_{mn}-e^{-r_1 a_{mn} z}B_{mn})+\frac{n\pi}{b}\frac{R_2}{\alpha_{mn}}(e^{r_2 a_{mn} z}C_{mn}-e^{-r_2 a_{mn} z}D_{mn})-\\&\frac{bm}{an}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn} z}E_{mn}-\frac{bm}{an}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn} z}F_{mn}]\\w(x,y,z)=&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(e^{r_1 a_{mn} z}A_{mn}+e^{-r_1 a_{mn} z}B_{mn}+e^{r_2 a_{mn} z}C_{mn}+e^{-r_2 a_{mn} z}D_{mn})\end{aligned} \tag{14}$$

in which

$$R_1 = \frac{\dfrac{C_{11}+C_{12}}{2(C_{13}+C_{44})}[\dfrac{(C_{11}+C_{12})C_{33}-2(C_{13}+C_{44})^2}{(C_{11}+C_{12})C_{44}}r_1^{\,2}-1]}{r_1(r_1^{\,2}-\dfrac{C_{11}-C_{12}}{2C_{44}})}$$

$$R_2 = \frac{\dfrac{C_{11}+C_{12}}{2(C_{13}+C_{44})}[\dfrac{(C_{11}+C_{12})C_{33}-2(C_{13}+C_{44})^2}{(C_{11}+C_{12})C_{44}}r_2^{\,2}-1]}{r_2(r_2^{\,2}-\dfrac{C_{11}-C_{12}}{2C_{44}})} \tag{15}$$

Substituting Eq. (14) into Eq. (1), one has

$$\sigma_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\{$$
$$[C_{13}r_1 a_{mn} - \frac{\dfrac{m^2\pi^2}{a^2}C_{11}+\dfrac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_1](e^{r_1 a_{mn}z}A_{mn} - e^{-r_1 a_{mn}z}B_{mn}) +$$
$$[C_{13}r_2 a_{mn} - \frac{\dfrac{m^2\pi^2}{a^2}C_{11}+\dfrac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_2](e^{r_2 a_{mn}z}C_{mn} - e^{-r_2 a_{mn}z}D_{mn}) +$$
$$\frac{m\pi}{a}(C_{12}-C_{11})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}E_{mn} + \frac{m\pi}{a}(C_{12}-C_{11})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}F_{mn}\}$$

$$\sigma_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\{$$
$$[C_{13}r_1 a_{mn} - \frac{\dfrac{m^2\pi^2}{a^2}C_{12}+\dfrac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_1](e^{r_1 a_{mn}z}A_{mn} - e^{-r_1 a_{mn}z}B_{mn}) +$$
$$[C_{13}r_2 a_{mn} - \frac{\dfrac{m^2\pi^2}{a^2}C_{12}+\dfrac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_2](e^{r_2 a_{mn}z}C_{mn} - e^{-r_2 a_{mn}z}D_{mn}) -$$
$$\frac{m\pi}{a}(C_{12}-C_{11})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}E_{mn} + \frac{m\pi}{a}(C_{12}-C_{11})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}F_{mn}\}$$

$$\sigma_z = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\{$$
$$[C_{33}r_1 a_{mn} - C_{13}a_{mn}R_1](e^{r_1 a_{mn}z}A_{mn} - e^{-r_1 a_{mn}z}B_{mn}) + [C_{33}r_2 a_{mn} - C_{13}a_{mn}R_2](e^{r_2 a_{mn}z}C_{mn} - e^{-r_2 a_{mn}z}D_{mn})\} \tag{16}$$

$$\tau_{xz} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[$$
$$\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})(e^{r_1 a_{mn}z}A_{mn} + e^{-r_1 a_{mn}z}B_{mn}) + \frac{m\pi}{a}(C_{44}+C_{44}\frac{R_2}{r_2})(e^{r_2 a_{mn}z}C_{mn} + e^{-r_2 a_{mn}z}D_{mn}) +$$
$$C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}\,a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}E_{mn} -$$
$$C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}\,a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}F_{mn}]$$

$$\tau_{yz} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}[$$

$$\frac{n\pi}{b}(C_{44} + C_{44}\frac{R_1}{r_1})(e^{r_1 a_{mn} z}A_{mn} + e^{-r_1 a_{mn} z}B_{mn}) + \frac{n\pi}{b}(C_{44} + C_{44}\frac{R_2}{r_2})(e^{r_2 a_{mn} z}C_{mn} + e^{-r_2 a_{mn} z}D_{mn}) -$$

$$\frac{bm}{an}C_{44}\sqrt{(C_{11} - C_{12})/2/C_{44}}\,a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}E_{mn} +$$

$$\frac{bm}{an}C_{44}\sqrt{(C_{11} - C_{12})/2/C_{44}}\,a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}F_{mn}]$$

$$\tau_{xy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}[$$

$$(C_{11} - C_{12})\frac{mn\pi^2}{a_{mn}ab}R_1(e^{r_1 a_{mn} z}A_{mn} - e^{-r_1 a_{mn} z}B_{mn}) + (C_{11} - C_{12})\frac{mn\pi^2}{a_{mn}ab}R_2(e^{r_2 a_{mn} z}C_{mn} - e^{-r_2 a_{mn} z}D_{mn}) +$$

$$\frac{C_{11} - C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2} - \frac{m^2\pi^2}{a^2})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}E_{mn} +$$

$$\frac{C_{11} - C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2} - \frac{m^2\pi^2}{a^2})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}z}F_{mn}]$$

where $A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}$ and $F_{mn}$ are the unknown coefficients and can be determined by the boundary conditions of the upper and lower surfaces of the plate.

## 3. UPPER AND LOWER SURFACE CONDITIONS

Consider the upper surface of the plate to be activated by the distributed load $q(x, y)$ and the lower surface of the plate to be free. The upper surface conditions of the plate can then be written as

$$\begin{aligned} l_1(x,y)\sigma_x + m_1(x,y)\tau_{yx} + n_1(x,y)\tau_{zx} &= 0 \\ m_1(x,y)\sigma_y + n_1(x,y)\tau_{zy} + l_1(x,y)\tau_{xy} &= 0 \\ n_1(x,y)\sigma_z + l_1(x,y)\tau_{xz} + m_1(x,y)\tau_{yz} &= q(x,y) \end{aligned} \tag{17}$$

where

$$l_1(x,y) = -\frac{\partial f_1(x,y)}{\partial x}/Q_1,\; m_1(x,y) = -\frac{\partial f_1(x,y)}{\partial y}/Q_1,\; n_1(x,y,z) = -1/Q_1 \tag{18}$$

in which

$$Q_1 = \sqrt{1 + [\frac{\partial f_1(x,y)}{\partial x}]^2 + [\frac{\partial f_1(x,y)}{\partial y}]^2} \tag{19}$$

And the lower surface conditions of the plate can be written as

$$\begin{aligned} l_2(x,y)\sigma_x + m_2(x,y)\tau_{yx} + n_2(x,y)\tau_{zx} &= 0 \\ m_2(x,y)\sigma_y + n_2(x,y)\tau_{zy} + l_2(x,y)\tau_{xy} &= 0 \\ n_2(x,y)\sigma_z + l_2(x,y)\tau_{xz} + m_2(x,y)\tau_{yz} &= 0 \end{aligned} \tag{20}$$

where

$$l_2(x,y) = -\frac{\partial f_2(x,y)}{\partial x}/Q_2,\; m_2(x,y) = -\frac{\partial f_2(x,y)}{\partial y}/Q_2,\; n_2(x,y,z) = 1/Q_2 \tag{21}$$

in which

$$Q_2 = \sqrt{1 + [\frac{\partial f_2(x,y)}{\partial x}]^2 + [\frac{\partial f_2(x,y)}{\partial y}]^2} \tag{22}$$

## 4. UNKNOWN COEFFICIENTS

Substituting Eq. (16) into Eqs. (17) and (20) and making the expansion of double Fourier sinusoidal series to the upper and lower surface equations, one has

$$\begin{aligned}
&\int_0^a \int_0^b [l_1(x,y)\sigma_x + m_1(x,y)\tau_{yx} + n_1(x,y)\tau_{zx}]\sin(\frac{i\pi x}{a})\sin(\frac{j\pi y}{b})dydx = 0 \\
&\int_0^a \int_0^b [m_1(x,y)\sigma_y + n_1(x,y)\tau_{zy} + l_1(x,y)\tau_{xy}]\sin(\frac{i\pi x}{a})\sin(\frac{j\pi y}{b})dydx = 0 \\
&\int_0^a \int_0^b [n_1(x,y)\sigma_z + l_1(x,y)\tau_{xz} + m_1(x,y)\tau_{yz}]\sin(\frac{i\pi x}{a})\sin(\frac{j\pi y}{b})dydx = \\
&\int_0^a \int_0^b q(x,y)\sin(\frac{i\pi x}{a})\sin(\frac{j\pi y}{b})dydx, \\
&\int_0^a \int_0^b [l_2(x,y)\sigma_x + m_2(x,y)\tau_{yx} + n_2(x,y)\tau_{zx}]\sin(\frac{i\pi x}{a})\sin(\frac{j\pi y}{b})dydx = 0 \\
&\int_0^a \int_0^b [m_2(x,y)\sigma_y + n_2(x,y)\tau_{zy} + l_2(x,y)\tau_{xy}]\sin(\frac{i\pi x}{a})\sin(\frac{j\pi y}{b})dydx = 0 \\
&\int_0^a \int_0^b [n_2(x,y)\sigma_z + l_2(x,y)\tau_{xz} + m_2(x,y)\tau_{yz}]\sin(\frac{i\pi x}{a})\sin(\frac{j\pi y}{b})dydx = 0
\end{aligned} \tag{23}$$

Truncating each series of terms up to $N$+1, one has the following algebraic equations in the matrix form of

$$\begin{bmatrix}
[A^{(1)}_{ijmn}] & [B^{(1)}_{ijmn}] & [C^{(1)}_{ijmn}] & [D^{(1)}_{ijmn}] & [E^{(1)}_{ijmn}] & [F^{(1)}_{ijmn}] \\
[A^{(2)}_{ijmn}] & [B^{(2)}_{ijmn}] & [C^{(2)}_{ijmn}] & [D^{(2)}_{ijmn}] & [E^{(2)}_{ijmn}] & [F^{(2)}_{ijmn}] \\
[A^{(3)}_{ijmn}] & [B^{(3)}_{ijmn}] & [C^{(3)}_{ijmn}] & [D^{(3)}_{ijmn}] & [E^{(3)}_{ijmn}] & [F^{(3)}_{ijmn}] \\
[A^{(4)}_{ijmn}] & [B^{(4)}_{ijmn}] & [C^{(4)}_{ijmn}] & [D^{(4)}_{ijmn}] & [E^{(4)}_{ijmn}] & [F^{(4)}_{ijmn}] \\
[A^{(5)}_{ijmn}] & [B^{(5)}_{ijmn}] & [C^{(5)}_{ijmn}] & [D^{(5)}_{ijmn}] & [E^{(5)}_{ijmn}] & [F^{(5)}_{ijmn}] \\
[A^{(6)}_{ijmn}] & [B^{(6)}_{ijmn}] & [C^{(6)}_{ijmn}] & [D^{(6)}_{ijmn}] & [E^{(6)}_{ijmn}] & [F^{(6)}_{ijmn}]
\end{bmatrix} \cdot \begin{bmatrix} \{A_{mn}\} \\ \{B_{mn}\} \\ \{C_{mn}\} \\ \{D_{mn}\} \\ \{E_{mn}\} \\ \{F_{mn}\} \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{0\} \\ \{q_{ij}\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \tag{24}$$

The elements in the above equation, which have been numerically evaluated by use of the piecewise Gaussian quadrature, given in the Appendix. Solving Eq. (24), the coefficients $A_{11},\ldots,A_{NN}, B_{11},\ldots,B_{NN}, C_{11},\ldots,C_{NN}, D_{11},\ldots,D_{NN}$, $E_{11},\ldots,E_{NN}$ and $F_{11},\ldots,F_{NN}$ can be uniquely determined. Substituting these coefficients into equations (14) and (16), the displacement and stress distributions of the plate can be obtained.

## 5. CONVERGENCE STUDIES

In the following studies, all the numerical computations were performed in double precision and the piecewise Gaussian quadrature was used numerically to evaluate each element in Eq. (24). The material properties (unit: $10^{10}$N.m$^{-2}$) of transversely isotropic rectangular plates are fixed at $C_{11}$=7.1, $C_{12}$=3.0, $C_{13}$=3.0, $C_{33}$=7.0, $C_{44}$=2.0, otherwise stated. In order to verify the accuracy of the proposed method, the convergence of the present solutions is studied first. Figure 2 shows a wedge-shaped plate with a linearly varying lower surface in the $y$ direction, where $H$ is the smallest thickness and $H_1$ is the biggest thickness of the plate. The upper surface of the plate is horizontal and subjected to a uniform load $q$ in the

$z$ direction. The sizes of the plate are $a$=$b$=10m and $H$=1m. Four different series terms $N$=12,16,20,24 have been checked, respectively. Table 1 gives the convergence of the stresses and displacements at the point $x = 3, y = 4, z = 0.1$. The stresses and displacements with two depth ratios $H_1/H$=1.5, 2.0 have been considered, respectively. It can be seen from Table 1 that the numerical results between $N$=20 and $N$=24 have a good agreement. The results from $N$=20 are accurate up to the third significant digit for both cases. The maximum relative errors of stresses between $N$=16 and $N$=20 are no more than 0.8%. This indicates the rapid convergence of the proposed method. Therefore, all the series terms are fixed at $N$=20 in the following numerical calculations. It should be mentioned that the usable number of the series terms in numerical calculations is limited, relating to the effective digit of the computer. Overly increasing the number of the series terms may lead to ill-conditioned results.

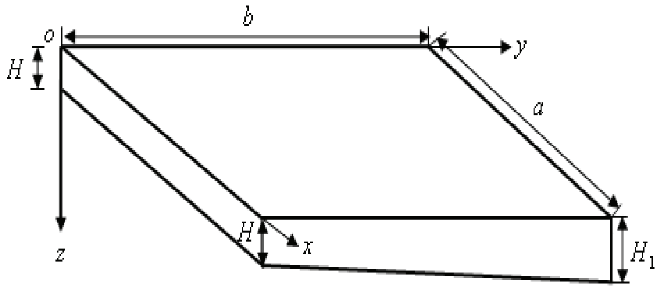



Fig. 2. Transversely isotropic rectangular plate with linearly varying lower surface

Table 1. The convergence of stresses and displacements of transversely isotropic rectangular plate ($a = 10m, b = 10m, H = 1m$) with linearly varying lower surface in the $y$ direction

| $H_1/H$ | $N$ | $\sigma_x/q$ | $\tau_{xy}/q$ | $uC_{44}/(qH)$ | $wC_{44}/(qH)$ |
|---|---|---|---|---|---|
| 1.5 | 12 | -14.1 | 1.43 | 19.9 | 201 |
| | 16 | -14.3 | 1.46 | 20.1 | 202 |
| | 20 | -14.4 | 1.47 | 20.2 | 202 |
| | 24 | -14.4 | 1.47 | 20.2 | 202 |
| 2.0 | 12 | -11.4 | 1.04 | 16.1 | 132 |
| | 16 | -11.5 | 1.06 | 16.2 | 133 |
| | 20 | -11.6 | 1.07 | 16.3 | 133 |
| | 24 | -11.6 | 1.07 | 16.3 | 133 |

## 6. COMPARISON STUDIES

A 3-D finite element simulation using the ANSYS software has been carried out to verify the correctness of the present three-dimensional elasticity solution for varying thickness plates. In the finite element analysis, the plate is modelled by the Solid-65 elements. The rectangular plate with parabolic convex lower surface in the $y$ direction is considered, as shown in Fig. 3. The upper surface of the plate is horizontal and subjected to the uniform load $q$ in the $z$ direction. The size of the plate is $a$=12m, $b$=10m, $H$=1m. The depth ratio of the plate is $H_1/H$=1.5. The stress and displacement distributions in the thickness direction along the line $x$=$a$/2, $z$=$H$/5 are given in Fig. 4. It can be seen from Fig. 4 that the present solutions agree with the FE solutions. It should be mentioned that the present three-dimensional analysis is based on the small-strain linear elasticity theory which does not rely on any hypotheses involving the kinematics of deformation. It is well known that both the classical plate theory and the FE approach predict the higher plate stiffness due to some imposed assumptions having been used. Therefore, it can be observed from Fig. 4 that the present static deflections are consistently larger than the FE solutions.

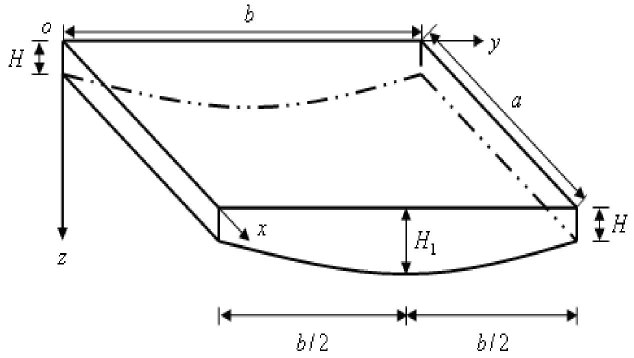
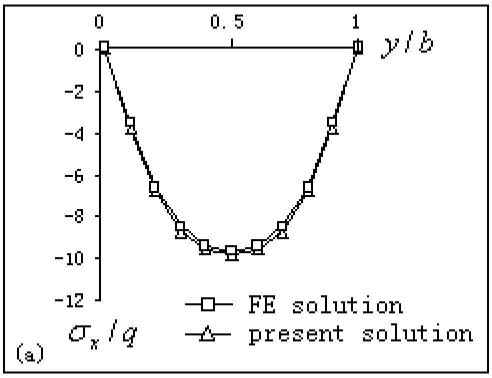



Fig. 3. Transversely isotropic rectangular plate with parabolic convex lower surface in the $y$ direction



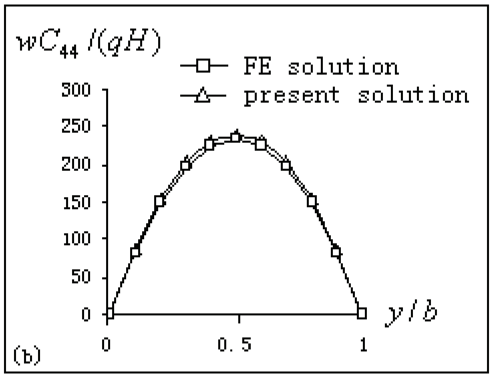




Fig. 4. The comparison of the three-dimensional elasticity solutions with the FE solutions for the plate with parabolic convex lower surface in the $y$ direction:
(a) $\sigma_x / q$, (b) $wC_{44}/(qH)$

## 7. NUMERICAL EXAMPLES

Consider a rectangular plate with a parabolic concave lower surface in the $y$ direction, as shown in Fig. 5. The upper surface of the plate is horizontal and subjected to the uniform load $q$ in the $z$ direction. The sizes of the plate are $a$=$b$=10m, $H$=2m. The stresses and displacements with three different depth ratios $H_1/H$=0.7,0.8,0.9 have been taken. The distributions of displacement component $w$ and stress component

$\sigma_y$ across the plate thickness along the line $x = a/2, y = b/2$ are displayed in Fig. 6. It can be seen from this figure that the maximum values of displacement component $w$ and stress component $\sigma_y$ remarkably decrease with the increase of the depth ratio $H_1/H$.

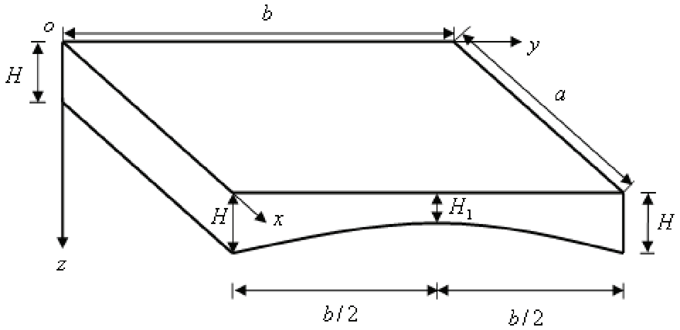



Fig. 5. Transversely isotropic rectangular plates with parabolic concave lower surface in the $y$ direction

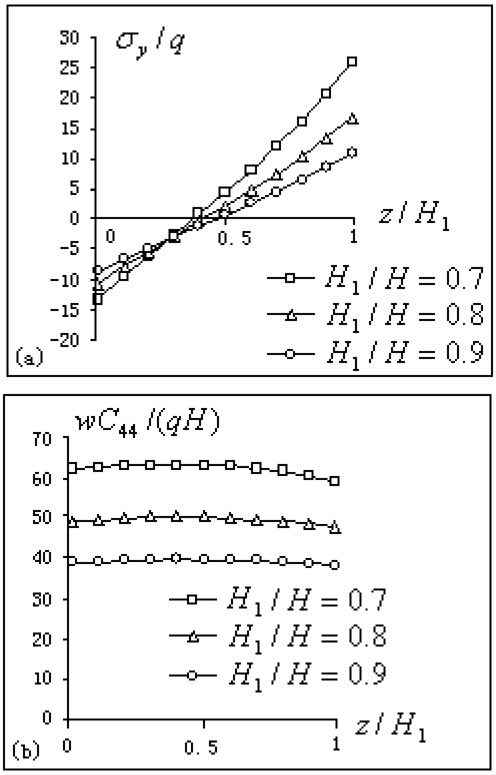



Fig. 6. The stress and displacement distributions on the line $x = a/2, y = b/2$ with different depth ratios: (a) $\sigma_y/q$ (b) $wC_{44}/(qH)$

## 8. CONCLUSION

The three-dimensional elasticity solutions of transversely isotropic rectangular plates with varying thickness have been studied. The general expressions for the displacements and stresses, which exactly satisfy the governing differential equations and the simply supported boundary conditions at four edges of the plate, are deduced. The unknown coefficients in the solutions are approximately determined by using the double Fourier sinusoidal series expansion to the upper surface and lower surface equations of the plate. The method is suitable for the analysis of stress and displacement distributions of continuously varying thickness plates. The validity and high accuracy of the proposed method has been verified by the convergence and comparison studies.

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**Appendix:** Elements in Eq. (24)

$$A_{ijmn}^{(1)}=\int_0^a\int_0^b\{l_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_1a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{11}+\frac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_1]+$$

$$m_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_1+$$

$$n_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_1a_{mn}f_1(x,y)}dxdy$$

$$B_{ijmn}^{(1)}=\int_0^a\int_0^b\{-l_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_1a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{11}+\frac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_1]-$$

$$m_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_1+$$

$$n_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_1a_{mn}f_1(x,y)}dxdy,$$

$$C_{ijmn}^{(1)}=\int_0^a\int_0^b\{l_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_2a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{11}+\frac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_2]+$$

$$m_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_2+$$

$$n_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_2a_{mn}f_1(x,y)}dxdy,$$

$$D_{ijmn}^{(1)}=\int_0^a\int_0^b\{-l_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_2a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{11}+\frac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_2]-$$

$$m_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_2+$$

$$n_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_1a_{mn}f_1(x,y)}dxdy,$$

$$E_{ijmn}^{(1)}=\int_0^a\int_0^b\{l_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{12}-C_{11})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}+$$

$$m_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{C_{11}-C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2}-\frac{m^2\pi^2}{a^2})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}+$$

$$n_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$F_{ijmn}^{(1)}=\int_0^a\int_0^b\{l_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{12}-C_{11})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}+$$

$$m_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{C_{11}-C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2}-\frac{m^2\pi^2}{a^2})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}-$$

$$n_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$A_{ijmn}^{(2)}=\int_0^a\int_0^b\{m_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_1a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{12}+\frac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_1]+$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_1+$$
$$n_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_1a_{mn}f_1(x,y)}dxdy\,,$$

$$B_{ijmn}^{(2)}=\int_0^a\int_0^b\{-m_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_1a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{12}+\frac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_1]-$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_1+$$
$$n_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_1a_{mn}f_1(x,y)}dxdy\,,$$

$$C_{ijmn}^{(2)}=\int_0^a\int_0^b\{m_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_2a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{12}+\frac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_2]+$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_2+$$
$$n_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_2a_{mn}f_1(x,y)}dxdy\,,$$

$$D_{ijmn}^{(2)}=\int_0^a\int_0^b\{-m_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_2a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{12}+\frac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_2]-$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_2+$$
$$n_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_2a_{mn}f_1(x,y)}dxdy\,,$$

$$E_{ijmn}^{(2)}=\int_0^a\int_0^b\{-m_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{12}-C_{11})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}+$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{C_{11}-C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2}-\frac{m^2\pi^2}{a^2})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}-$$
$$n_1(x,y)\frac{bm}{an}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$F_{ijmn}^{(2)}=\int_0^a\int_0^b\{m_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{12}-C_{11})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}+$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{C_{11}-C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2}-\frac{m^2\pi^2}{a^2})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}+$$

$$n_1(x,y)\frac{bm}{an}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$A_{ijmn}^{(3)}=\int_0^a\int_0^b\{n_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{33}r_1a_{mn}-C_{13}a_{mn}R_1]+$$
$$m_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_1}{r_1})+$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_1a_{mn}f_1(x,y)}dxdy,$$

$$B_{ijmn}^{(3)}=\int_0^a\int_0^b\{-n_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{33}r_1a_{mn}-C_{13}a_{mn}R_1]+$$
$$m_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_1}{r_1})+$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_1a_{mn}f_1(x,y)}dxdy,$$

$$C_{ijmn}^{(3)}=\int_0^a\int_0^b\{n_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{33}r_2a_{mn}-C_{13}a_{mn}R_2]+$$
$$m_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_2}{r_2})+$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_2a_{mn}f_1(x,y)}dxdy,$$

$$D_{ijmn}^{(3)}=\int_0^a\int_0^b\{-n_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{33}r_2a_{mn}-C_{13}a_{mn}R_2]+$$
$$m_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_2}{r_2})+$$
$$l_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_2a_{mn}f_1(x,y)}dxdy,$$

$$E_{ijmn}^{(3)}=\int_0^a\int_0^b\{l_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}+$$
$$m_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{bm}{an}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$F_{ijmn}^{(3)}=\int_0^a\int_0^b\{-l_1(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}+$$
$$m_1(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{bm}{an}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_1(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$A_{ijmn}^{(4)}=\int_0^a\int_0^b\{l_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_1a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{11}+\frac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_1]+$$
$$m_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_1+$$

$$n_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_1 a_{mn} f_2(x,y)}dxdy$$

$$B_{ijmn}^{(4)}=\int_0^a\int_0^b\{-l_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_1a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{11}+\frac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_1]-$$

$$m_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_1+$$

$$n_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_1 a_{mn} f_2(x,y)}dxdy\,,$$

$$C_{ijmn}^{(4)}=\int_0^a\int_0^b\{l_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_2a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{11}+\frac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_2]+$$

$$m_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_2+$$

$$n_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_2 a_{mn} f_2(x,y)}dxdy\,,$$

$$D_{ijmn}^{(4)}=\int_0^a\int_0^b\{-l_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_2a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{11}+\frac{n^2\pi^2}{b^2}C_{12}}{a_{mn}}R_2]-$$

$$m_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_2+$$

$$n_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_1 a_{mn} f_2(x,y)}dxdy\,,$$

$$E_{ijmn}^{(4)}=\int_0^a\int_0^b\{l_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{12}-C_{11})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}+$$

$$m_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{C_{11}-C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2}-\frac{m^2\pi^2}{a^2})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}+$$

$$n_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$F_{ijmn}^{(4)}=\int_0^a\int_0^b\{l_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{12}-C_{11})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}+$$

$$m_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{C_{11}-C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2}-\frac{m^2\pi^2}{a^2})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}-$$

$$n_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$A_{ijmn}^{(5)}=\int_0^a\int_0^b\{m_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_1a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{12}+\frac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_1]+$$

$$l_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_1+$$

$$n_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_1 a_{mn} f_2(x,y)}dxdy,$$

$$B_{ijmn}^{(5)}=\int_0^a\int_0^b\{-m_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_1a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{12}+\frac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_1]-$$
$$l_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_1+$$
$$n_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_1 a_{mn} f_2(x,y)}dxdy,$$

$$C_{ijmn}^{(5)}=\int_0^a\int_0^b\{m_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_2a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{12}+\frac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_2]+$$
$$l_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_2+$$
$$n_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_2 a_{mn} f_2(x,y)}dxdy,$$

$$D_{ijmn}^{(5)}=\int_0^a\int_0^b\{-m_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{13}r_2a_{mn}-\frac{\frac{m^2\pi^2}{a^2}C_{12}+\frac{n^2\pi^2}{b^2}C_{11}}{a_{mn}}R_2]-$$
$$l_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}(C_{11}-C_{12})\frac{mn\pi^2}{a_{mn}ab}R_2+$$
$$n_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_2 a_{mn} f_2(x,y)}dxdy,$$

$$E_{ijmn}^{(5)}=\int_0^a\int_0^b\{-m_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{12}-C_{11})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}+$$
$$l_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{C_{11}-C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2}-\frac{m^2\pi^2}{a^2})e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}-$$
$$n_2(x,y)\frac{bm}{an}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$F_{ijmn}^{(5)}=\int_0^a\int_0^b\{m_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{12}-C_{11})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}+$$
$$l_2(x,y)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{C_{11}-C_{12}}{2}\frac{b}{n\pi}(\frac{n^2\pi^2}{b^2}-\frac{m^2\pi^2}{a^2})e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}+$$
$$n_2(x,y)\frac{bm}{an}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$A_{ijmn}^{(6)}=\int_0^a\int_0^b\{n_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{33}r_1a_{mn}-C_{13}a_{mn}R_1]+$$
$$m_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_1}{r_1})+$$

$$l_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_1 a_{mn} f_2(x,y)}dxdy\,,$$

$$B_{ijmn}^{(6)}=\int_0^a\int_0^b\{-n_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{33}r_1a_{mn}-C_{13}a_{mn}R_1]+$$

$$m_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_1}{r_1})+$$

$$l_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_1}{r_1})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_1 a_{mn} f_2(x,y)}dxdy\,,$$

$$C_{ijmn}^{(6)}=\int_0^a\int_0^b\{n_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{33}r_2a_{mn}-C_{13}a_{mn}R_2]+$$

$$m_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_2}{r_2})+$$

$$l_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{r_2 a_{mn} f_2(x,y)}dxdy\,,$$

$$D_{ijmn}^{(6)}=\int_0^a\int_0^b\{-n_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}[C_{33}r_2a_{mn}-C_{13}a_{mn}R_2]+$$

$$m_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{n\pi}{b}(C_{44}+C_{44}\frac{R_2}{r_2})+$$

$$l_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{m\pi}{a}(C_{44}+C_{44}\frac{R_2}{r_2})\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}e^{-r_2 a_{mn} f_2(x,y)}dxdy\,,$$

$$E_{ijmn}^{(6)}=\int_0^a\int_0^b\{l_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}+$$

$$m_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{bm}{an}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$F_{ijmn}^{(6)}=\int_0^a\int_0^b\{-l_2(x,y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}+$$

$$m_2(x,y)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\frac{bm}{an}C_{44}\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}e^{-\sqrt{(C_{11}-C_{12})/2/C_{44}}a_{mn}f_2(x,y)}\}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dxdy$$

$$q_{ij}=\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$