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EFFECT OF CONVECTIVE TERM SUPPRESSION IN NUMERICAL SIMULATION OF TRANS-CRITICAL OPEN CHANNEL FLOW*

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Abstract– The implementation of boundary conditions in sub-critical and supercritical flow is quite different when using characteristics leading to programming difficulties with the associated numerical schemes. For supercritical flow, the de Saint Venant equations require two upstream boundary conditions and no downstream condition, whereas sub-critical flow requires one upstream and one downstream condition. The literature contains many approaches to accommodate both super- and sub-critical flows. Reducing or suppressing the convective term is one of the common methods which allows the same numerical scheme to be used for both regimes. In this paper, the impact of suppressing the convective term on the solution is investigated using the Method of Characteristics (MOC). A set of numerical experiments are carried out for this purpose using the commercial software MIKE11, and the results are compared and contrasted with MOC. Results show that significant changes in computed water depths occur in some situations by suppressing the convective term. In conclusion, in some cases the solution algorithm is significantly affected by this approximation. Also, since recent advances in numerical modeling of trans-critical flow are superior, this approximation should gradually be removed from the numerical simulation of open channel flow.

Keywords – Method of characteristics (MOC), flow regime, convective term, de Saint-Venant equations, trans-critical flow, MIKE11

1. INTRODUCTION

The Saint Venant equations are widely used for the numerical simulation of free surface flow in open channels and rivers. These equations, which can be obtained by integrating the Navier-Stokes equations over the cross section, simulate unsteady spatially varied flow in non-prismatic open channels.

In natural streams or artificial channels, there are many occasions where trans-critical flow regime occurs. Trans-critical flow is the term that describes the existence of supercritical and sub-critical flows within the considered domain under unsteady conditions. Trans-critical flow occurs frequently in irrigation canals and other man-made structures found in drainage networks. Such flow is most often encountered on steep slopes and can arise in naturally formed rivers in mountainous areas.

It can easily be shown that in supercritical flow, unlike sub-critical flow, the state of flow is not influenced by downstream conditions and so only upstream boundary conditions are implemented to simulate these flows, while in sub-critical flow the state of flow is affected by both upstream and downstream conditions. This essential difference between sub-critical and supercritical flows causes programming difficulties and a higher computational cost in modeling trans-critical flow (where the flow regime changes from sub-critical to supercritical and vice versa). In general, most unsteady flow solution algorithms become unstable when the flow passes through critical depth. Therefore, many efforts have been made to deal with numerical simulation of this type of flow in open channels [1-4].

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The convective term in the momentum equation can significantly alter the manner in which boundary conditions are applied [5]. Indeed, by eliminating and/or gradually reducing this term (also called suppression of convective term), supercritical flow behaves numerically like subcritical flow in terms of boundary conditions. Using this technique, programming difficulties could be overcome [6, 7]. The consequences of the suppression of the convective term have only very recently received attention in hydraulic literature [8, 9]. Kutija investigated the impact of eliminating and/or gradually reducing the convective terms for various kinds of momentum equations in a channel where supercritical flow occurs over the entire channel length [8]. She concluded that the full reduction of the convective term should be avoided and it is better to reduce the convective term partially. She did not investigate the impact of such suppression on subsequent flow regime where supercritical flow changes to sub-critical flow and vice versa (i.e., trans-critical flow). In order to solve the stability problem for a trans-critical flow regime system, Fread *et al*. [1] took a totally different viewpoint and developed a methodology called the "Local Partial Inertia" (LPI) technique. The LPI method has been adapted to HEC-RAS as an option for solving mixed flow regime problems when using the unsteady flow analysis portion of HEC-RAS. This methodology applies a reduction factor to the two inertia terms in the momentum equation as the Froude number goes towards 1.0. Some efforts have been made to find numerical schemes for trans-critical flow without suppressing the convective term. A new method called NewC is presented for the simulation of trans-critical flow without requiring any changes to the governing equations [9].

In this paper, the impact of the convective term suppression on the solution of the Saint-Venant equations for the case of trans-critical flow will be investigated and highlighted. A hypothetical channel is used for comparison purposes, the solution of which is available using Preissmann's four-point implicit scheme [10]. At first, the complete form of the Saint-Venant equations (fully dynamic) is solved using the Method of Characteristics (MOC). Many other numerical methods could also be used for this purpose, however, the method of characteristics was used as it still deserves special attention and understanding for the following reasons: First of all, the close relationship between physical and mathematical properties makes this method a basic concept and tool in analyzing the complex problem of unsteady trans-critical flow regime. Also, this method is generally considered to be more accurate than other methods and is sometimes used to provide a benchmark for comparisons. Furthermore, it seems to handle rapid changing flows more effectively than other methods [11]. Then, numerical results obtained by MOC are compared with MIKE11 results which incorporate convective term suppression.

2. THE ROLE OF CONVECTIVE TERM ON BOUNDARY CONDITIONS IN SAINT-VENANT EQUATIONS

In writing the Saint-Venant equations there are two possible alternatives for the inclusion of kinematic properties (e.g., discharge or velocity) and three for geometric properties (e.g., depth, flow area and stage). Hence, Saint-Venant equations can be casted in six different forms in terms of the state variables involved. These forms, although mathematically equivalent, have different numerical characteristics. Lai *et al*. [10] showed that the choice of state variables is the most important factor affecting how well the equations conserve mass. In this paper, discharge and flow area are taken as state variables. Using these variables, the Saint Venant equations become [12]:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q$$

$$\frac{\partial Q}{\partial t} + \beta \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial h}{\partial x} = gA(S_0 - S_f) + qu'$$

where A denotes flow area, Q= discharge, and h=depth of water with respect to the channel bottom (i.e., channel thalweg), q=lateral inflow, u'=x-component of velocity of lateral flow, S_0=channel bed slope and S_f = friction slope. Furthermore, in the momentum equation, $\frac{\partial}{\partial x}(\frac{Q^2}{A})$ is the convective term and β is a factor considered to evaluate the impact of convective term suppression. Upon ignoring the lateral inflow, expanding the convective term and expressing $\frac{\partial h}{\partial x}$ in terms of A, Saint Venant equations become:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{1}$$

$$\frac{\partial Q}{\partial t} + \beta [\frac{Q}{A}\frac{\partial Q}{\partial x} + Q\frac{\partial}{\partial x}(\frac{Q}{A})] + \frac{gA}{T}\frac{\partial A}{\partial x} = gA(S_0 - S_f) - gA\left.\frac{\partial h}{\partial x}\right|_{A=cte} \tag{2}$$

where T denotes top width. Assuming the prismatic channel, the last term in the momentum equation can be ignored. The friction slope evaluated using the Manning equation is:

$$S_f = n^2 Q^2 P^{\frac{4}{3}} A^{-\frac{10}{3}}, \quad P = P(A) \tag{3}$$

where n is the Manning coefficient and P is the wetted perimeter of the channel cross section. In order to investigate the effect of the convective term on the boundary conditions and direction of characteristic curves, the method of characteristics may be applied. By multiplying Eq. (1) by an unknown multiplier λ, adding it to Eq. (2) and rearranging the terms, the result is (assuming prismatic channel):

$$[\frac{\partial Q}{\partial t} + (2\beta\frac{Q}{A} + \lambda)\frac{\partial Q}{\partial x}] + \lambda[\frac{\partial A}{\partial t} + \frac{1}{\lambda}(-\beta\frac{Q^2}{A^2} + g\frac{A}{T})\frac{\partial A}{\partial x}] = gA(S_0 - S_f) \tag{4}$$

To transform the above partial differential equation into the corresponding ordinary differential equations (characteristics equations), λ must satisfy the following two equations:

$$2\beta\frac{Q}{A} + \lambda = \frac{dx}{dt} = \frac{1}{\lambda}(-\beta\frac{Q^2}{A^2} + g\frac{A}{T}) \tag{5}$$

which gives:

$$\lambda = -\beta\frac{Q}{A} \pm \sqrt{\beta(\beta-1)\frac{Q^2}{A^2} + g\frac{A}{T}} \quad \text{and} \quad \frac{dx}{dt} = \beta\frac{Q}{A} \pm \sqrt{\beta(\beta-1)\frac{Q^2}{A^2} + g\frac{A}{T}} \tag{6}$$

And characteristic equations become:

$$\frac{DQ}{Dt} + [-\beta\frac{Q}{A} \pm \sqrt{\beta(\beta-1)\frac{Q^2}{A^2} + g\frac{A}{T}}]\frac{DA}{Dt} = gA(S_0 - S_f) \tag{7}$$

$$\text{valid along} \qquad \frac{dx}{dt} = \beta\frac{Q}{A} \pm \sqrt{\beta(\beta-1)\frac{Q^2}{A^2} + g\frac{A}{T}} \tag{8}$$

Equation (6) denotes the slope of characteristic curves. In the typical form of the Saint Venant equations, β = 1, i.e., $\frac{dx}{dt} = \frac{Q}{A} \pm \sqrt{gA/T}$. Hence, the relative magnitude of flow velocity with respect to wave celerity determines the slope of the characteristics, which may be positive, zero or negative. Furthermore, based on the slope of the characteristic curves, three different states of fluid motion can be distinguished. For supercritical flow, the velocity of fluid ($\frac{Q}{A}$) is greater than wave celerity ($\sqrt{gA/T}$). In other words, both characteristics are positive. Therefore, the numerical value of state variables at any point P in the computational domain is affected by the situations upstream, and downstream conditions have no impact on flow regime. For critical flow condition, wave celerity is equal to flow velocity, resulting in a vertical slope for one characteristic line. Hence, negative characteristics become a vertical line for critical flow regime and state at a point in the domain is not influenced by the downstream flow condition. For sub-critical flow, the two characteristics have opposite signs and the state at a point is dependent on both downstream and upstream conditions [see Fig. 1]. In this figure, P corresponds to the intersection of the two characteristics, L refers to the intersection of the positive characteristic and previous time line, while R corresponds to the intersection of the negative characteristic and previous time line.

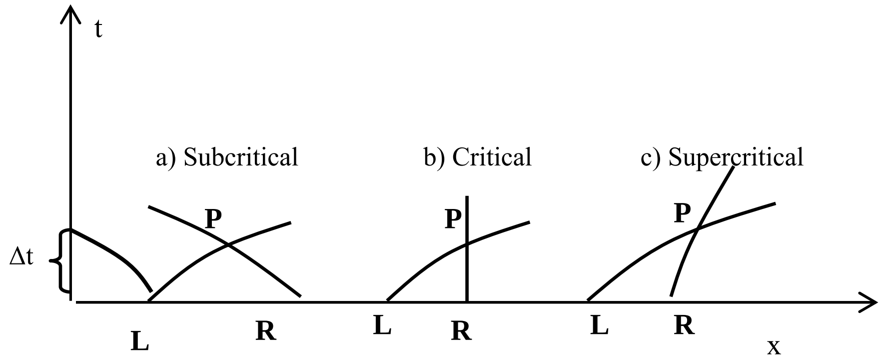


Fig. 1. Characteristics for sub-critical, critical and supercritical flow

It can easily be shown that the factor β which suppresses the convective term in the momentum equation can effectively control the slope of the characteristic curves, and hence, the required boundary conditions for supercritical flow. For example, by replacing β=0 (i.e., full reduction of the convective term) in Eq. (6), characteristics become: $\frac{dx}{dt} = \pm\sqrt{gA/T}$, which always have opposite signs independent of the flow regime. Another choice of β may gradually reduce the convective term. As an example, by taking $\beta = 1/F_n^b$ [where b is an exponent and F_n is Froude number] and replacing it in Eq. (6), the slope of characteristic curves after some mathematical manipulations becomes

$$\frac{dx}{dt} = \frac{Q}{AF_n^b}[1 \pm \sqrt{F_n^{2(b-1)} - F_n^b + 1}] \tag{9}$$

Equation (9) clearly shows that for all values of b greater than 2, the terms inside brackets have opposite signs for supercritical flow. Indeed, by suppressing the convective term in the momentum equation, supercritical flow behaves like sub-critical flow in terms of the required boundary conditions. The essential difference between sub-critical and supercritical flow in terms of boundary conditions leads to greater complexity and more computational effort. Accordingly, taking advantage of the concept of the suppression of the convective term, some prefer to use the same numerical scheme for both sub-critical and supercritical flow. In the next part of this paper, the effect of this simplification is investigated via numerical experiments.

3. NUMERICAL EXPERIMENTS

Figure 2 shows the hypothetical channel that is used for conducting the numerical experiments. The channel is divided into three different reaches. Reach one is considered for supercritical flow and reach three for sub-critical flow. Although slopes of the channel in reach one and three are equal, significant differences between frictional resistances leads to different flow regimes in two reaches. A small part of the channel, namely reach two, is designated for transition from supercritical to sub-critical flow. Flow discontinuity occurs as hydraulic jump in this reach. It should be noted that reach two is only considered for a change of the flow regime, and does not actually exist in the computational domain [10]. The channel has a uniform rectangular cross section having a 6 m width and a total length of 2000 m. It should be noted that this hypothetical channel was originally used by Lai *et al*. [10], having the advantage that in part of the spatial domain, the flow regime is supercritical (i.e., reach 1), followed by a sub-critical condition in reach 3. Furthermore, the finite difference solution of the flow is also available for comparison purposes [10]. The trust of this paper is to investigate the impact of convective term suppression on subsequent sub-critical flow downstream.

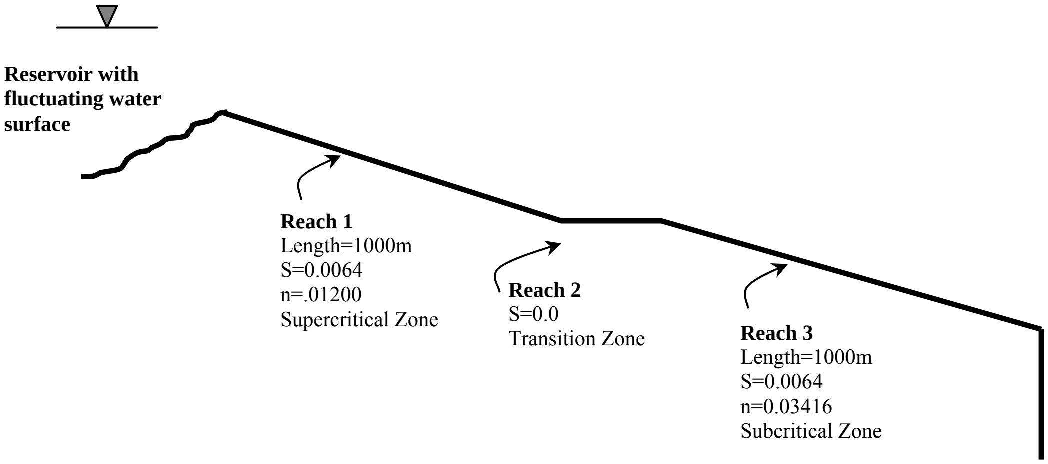


Fig. 2. Hypothetical channel used for numerical experiment, After Lai *et al.* [10]

The upstream of the channel is connected to a reservoir with a time varying water stage, imposing two boundary conditions in this section, which are the discharge area relationship and flow depth as a function of time (Flow area is simply $A(0, t) = bh(0, t)$) [Fig. 3]. Since the bottom slope in reach 1 is considered to be steep, the flow depth at the channel entrance will be critical. This critical depth could be converted to a discharge-flow area relationship easily by combining the energy equation and critical depth condition. Also, there is a drop at the end of the channel which similarly imposes a discharge–flow area relationship at the downstream point. Finally, the computed discharge as a function of time at the end of reach one is used as the upstream boundary condition for reach three.

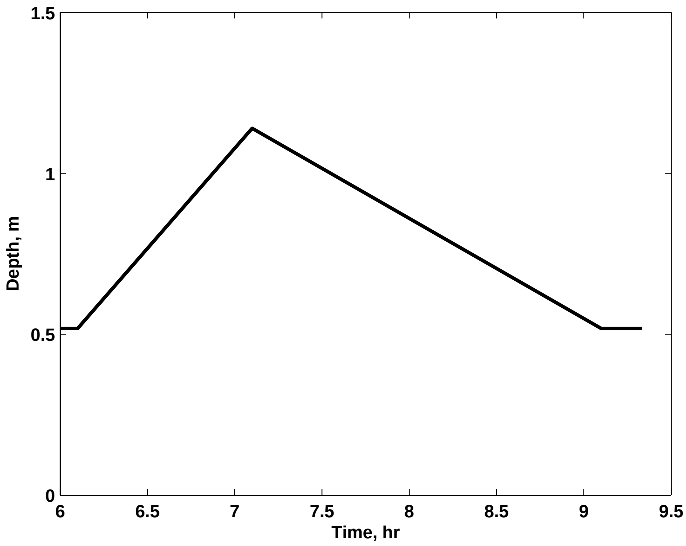


Fig. 3. Time series of flow depth versus time–Reservoir entrance

a) Method of characteristics as benchmark

Returning to Eqs. (7) and (8), the method of characteristics with the full inclusion of the convective term is used for the computation of the water surface profile and the impact assessment in the channel. This result will be used as a benchmark for the evaluation of the suppression of the convective term being implemented in a commercial program called MIKE11. Assuming β=1 (i.e., no reduction in convective term), Eq. (7) may be rewritten as:

$$\frac{DQ}{Dt} + \Psi^{\pm}(A,Q)\frac{DA}{Dt} = \Theta(A,S_f) \text{ valid along } \frac{dx}{dt} = \frac{Q}{A} \pm \sqrt{g\frac{A}{T}} = \Lambda^{\pm}(A,Q) \tag{10}$$

where

$$\Psi^{\pm}(A,Q) = -\frac{Q}{A} \pm \sqrt{g\frac{A}{T}} \text{ and } \Theta(A,S_f) = gA(S_0 - S_f) = gA(S_0 - n^2Q^2P^{\frac{4}{3}}A^{-\frac{10}{3}})$$

There are two common methods for discretizing the above equations, namely Grid of Characteristics (GC) and Specified-Time-Interval (STI) scheme [12]. Incorporating GC and using the second-order approximation for total derivatives gives [See Fig. 1 for location of points L, P and R.]:

$$Q_P - Q_R + \frac{1}{2}(\Psi_P^- + \Psi_R^-)(A_P - A_R) = \frac{1}{2}(\Theta_P + \Theta_R)(t_P - t_R) \tag{11}$$

$$\text{valid along} \qquad x_P - x_R = \frac{1}{2}(\Lambda_P^- + \Lambda_R^-)(t_P - t_R) \tag{12}$$

$$Q_P - Q_L + \frac{1}{2}(\Psi_P^+ + \Psi_L^+)(A_P - A_L) = \frac{1}{2}(\Theta_P + \Theta_L)(t_P - t_L) \tag{13}$$

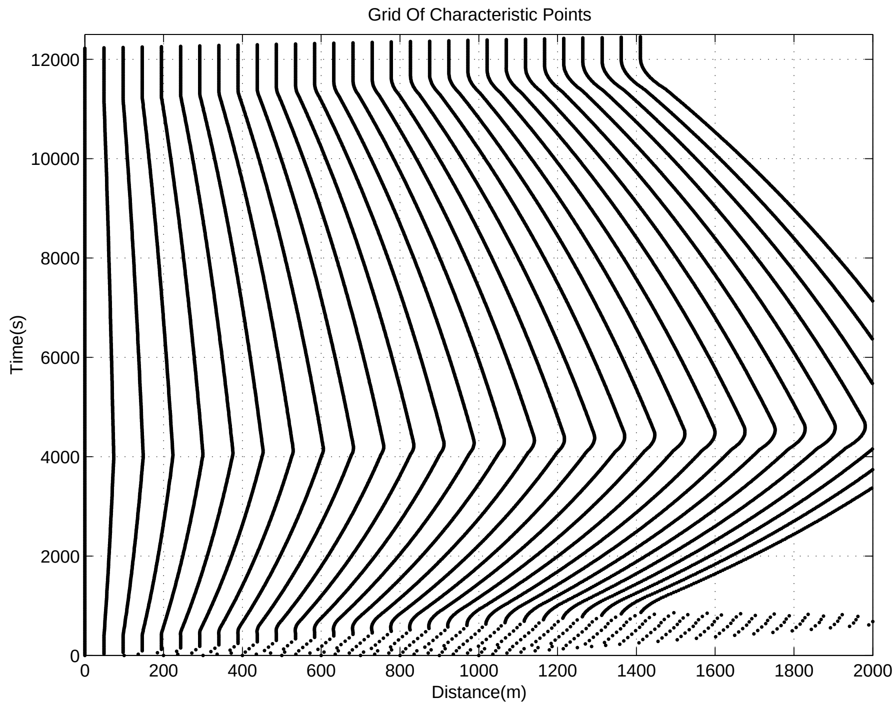
$$\text{valid along} \qquad x_P - x_L = \frac{1}{2}(\Lambda_P^+ + \Lambda_L^+)(t_P - t_L) \tag{14}$$

The GC scheme marches the solution ahead with the solved value t_p. The detail of this procedure for computing the values at the advanced time points is explained elsewhere [12, 13]. Traces of characteristic lines in the x-t plane for supercritical flow regime (about 60,000 points for this numerical experiment) is shown in Fig. 4.

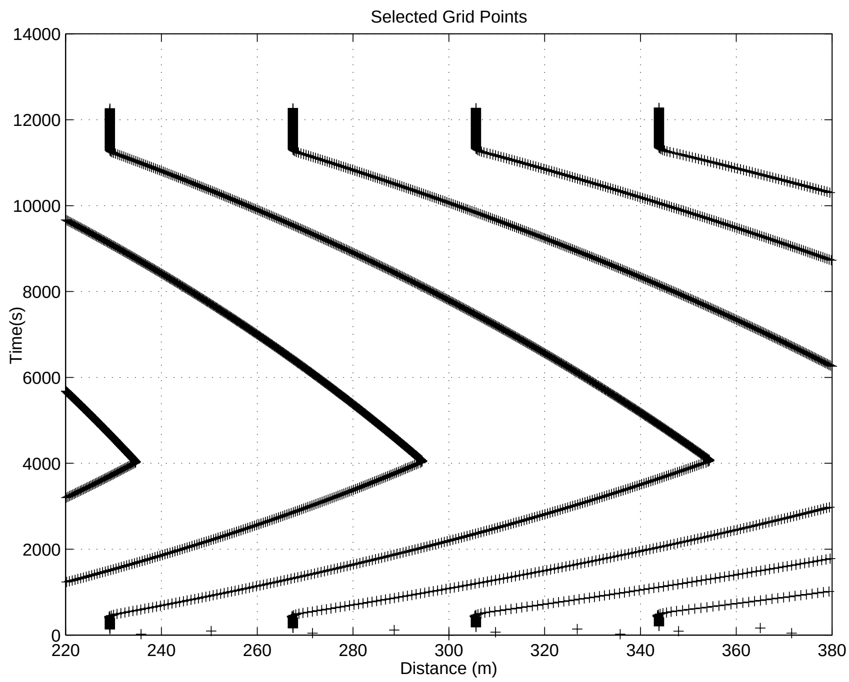
It should be noted that the time interval must be computed as a function of the space interval to ensure the stability of the solution. This may be achieved by satisfying the Courant number, i.e., setting the ratio of wave celerity to grid celerity equal to unity. Another useful method is to choose the first time interval so that the negative characteristic lines originating from x-axis intersects the time axis [11]. By implementing the above method, the discharge and flow area are computed in the channel. However, flow and depth [For the sake of comparison with MIKE11 results, depth (instead of flow area) is used in these figures as function of time are shown at selected locations of reach one and reach three in Figs. 5 and 7. The computed depths are similar to those obtained by Lai *et al*. [10] for the same numerical experiment and using the finite difference method.

b) Comparison with result obtained by suppressing convective terms using MIKE11

MIKE11 is a well-known software that has been extensively used for unsteady flow simulation in rivers and open channels. It solves almost the same equations as Eq. (2) and incorporates the concept of the suppression of the convective term. It has two options for the reduction of the convective term which allow the user to choose full reduction or partial reduction of the convective term [6, 14]. In this part, results obtained by the method of characteristics in the hypothetical channel are compared with MIKE11 results. As was previously noted, in the case of suppression, only two boundary conditions are required for the entire length of the channel, including reach 1 and reach 3. Accordingly, flow hydrograph at the start of reach one and the stage discharge relationship at the end of reach three are imposed as boundary conditions. By using this software, the flow is simulated in a hypothetical channel. Table 1 compares the depth and discharge solutions at peak value for various numerical schemes. A graphical representation of depth and discharge hydrographs can be seen in Figs. 5-8.

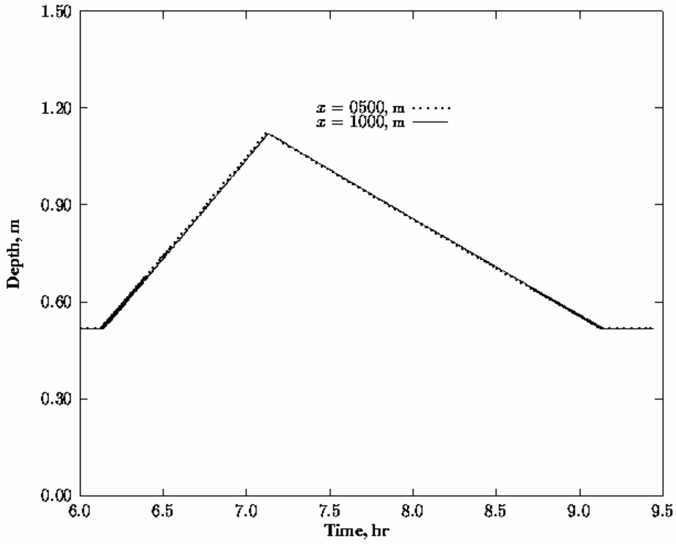


(a) The whole reach

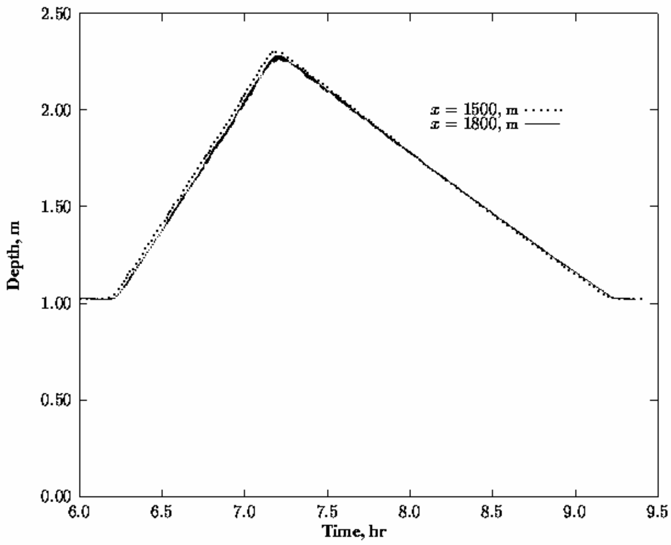


(b) Selected sub-reach

Fig. 4. Graphical representation of grid of characteristic points–Reach 1, supercritical flow



(a) Reach 1 at *x*=500 and *x*=1000 from the reservoir



(b) Reach 3 at *x*=1500 and *x*=1800 from the reservoir

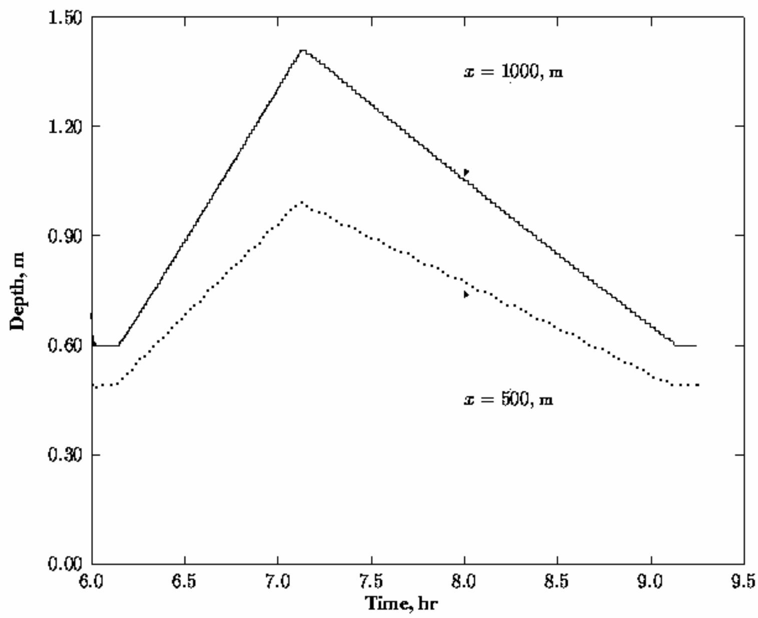
Fig. 5. Flow depth versus time—MOC results

Table 1. Comparison of results for various numerical schemes

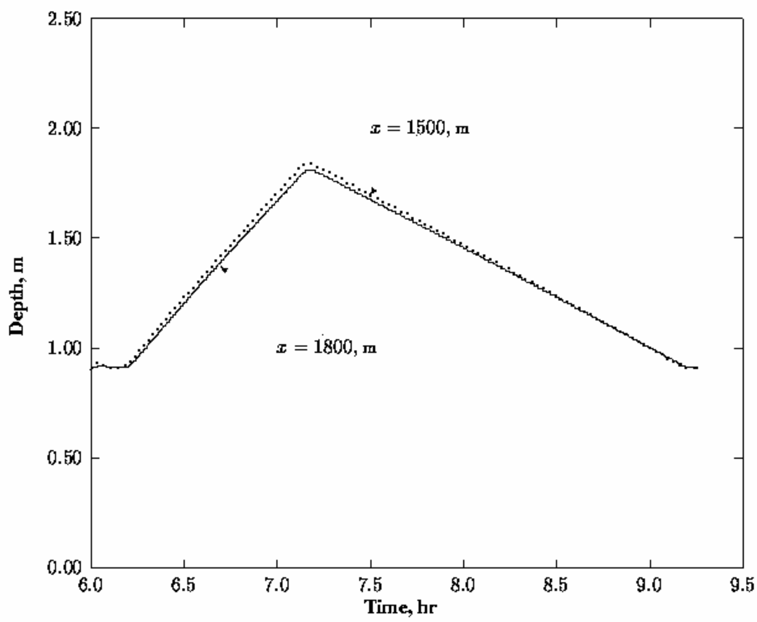
Distance† (m)	Maximum depth versus time Time (hr) Depth (m)	Maximum discharge versus time Time (hr) Discharge (m^3/s)
500	7:06:54 1.1216‡ 7:06:59 0.9900	7:06:54 39.182 7:06:52 39.200
1000	7:07:53 1.1207 7:08:14 1.4100	7:06:54 39.158 7:07:44 39.190
1500	7:10:34 2.3027 7:09:14 1.8400	7:10:03 38.705 7:08:59 39.010
1800	7:12:21 2.2824 7:10:14 1.8100	7:11:33 38.581 7:09:44 38.920

†Longitudinal distance is from reservoir entrance.

‡ Note: For every *x*, the first and second row of data correspond to the MOC and MIKE11 results, respectively.



(a) Reach 1 at *x*=500 and *x*=1000 from the reservoir

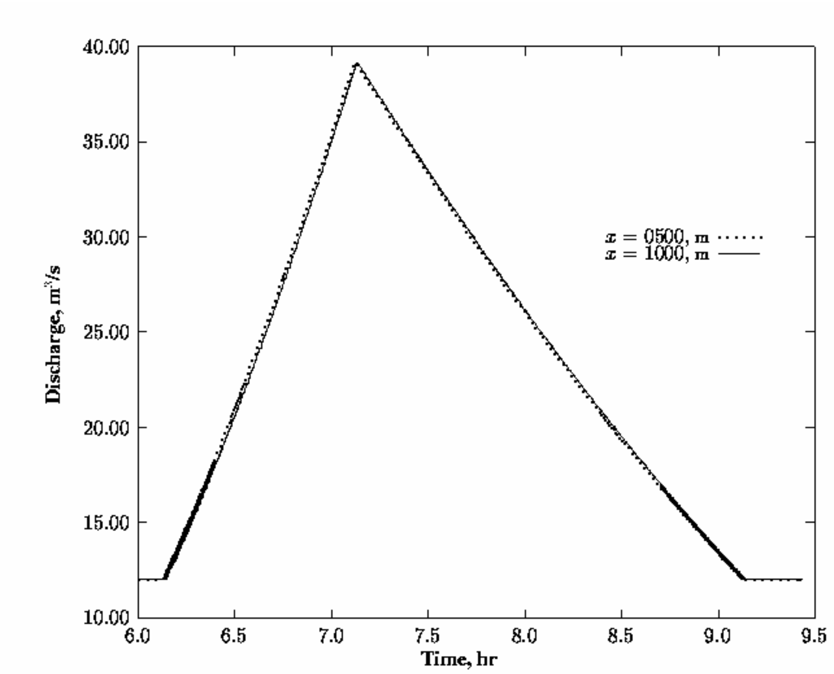


(b) Reach 3 at *x*=1500 and *x*=1800 from the reservoir

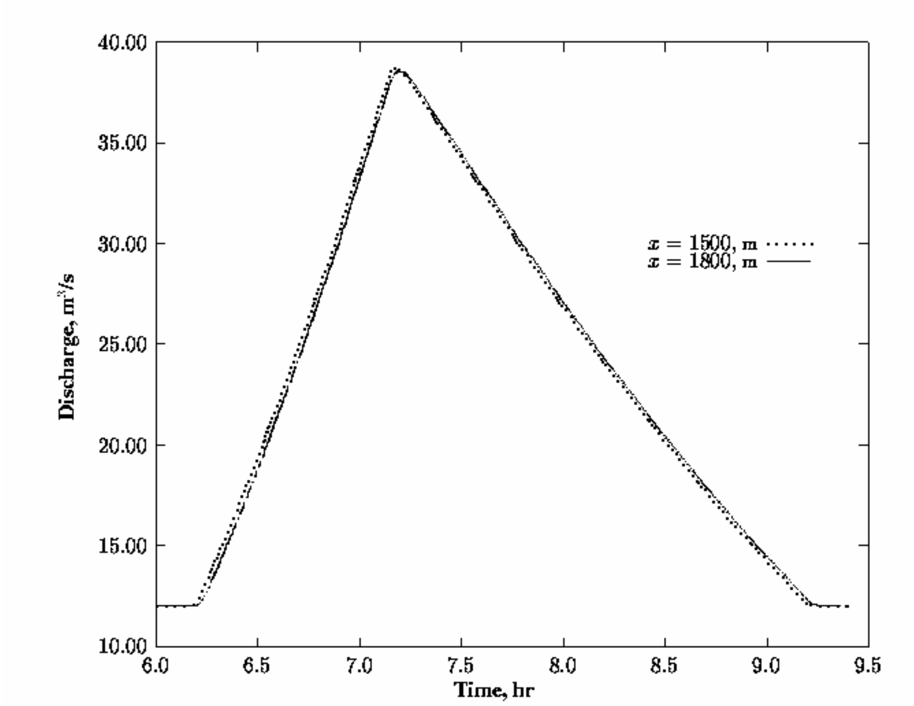
Fig. 6. Flow depth versus time—MIKE results

While the results obtained by Kutija [8] using models which were implementing the suppression of the convective term overestimated the water depth, the results obtained in this research partly confirmed previous findings, and also provided some new insights regarding the impact of such suppression on subsequent sub-critical flow downstream which had not been tried before. In reach 1, which is entirely dominated by supercritical flow, MIKE11 underestimated water depth at *x*=500 m by as much as 11.7%, and overestimated water depth at *x*=1000 m by as much as 25.8% compared to results obtained by MOC. In reach 3, which is entirely dominated by sub-critical flow, MIKE11 underestimated water depth at both

x=1500 m and x=1800 m sections as much as 20% compared to results obtained by MOC. As Table (1) clearly demonstrates, both numerical schemes simulate the timing of hydrographs (in particular, time to peak) quite well, implying no phase shift. The remarkable difference between depths for various numerical schemes is not perceivable in the case of discharge hydrographs in this study, due to a low range of discharge variation over the entire physical domain. Indeed, depth varies as much as 50% over the entire physical domain, while discharge varies only around 1.5%

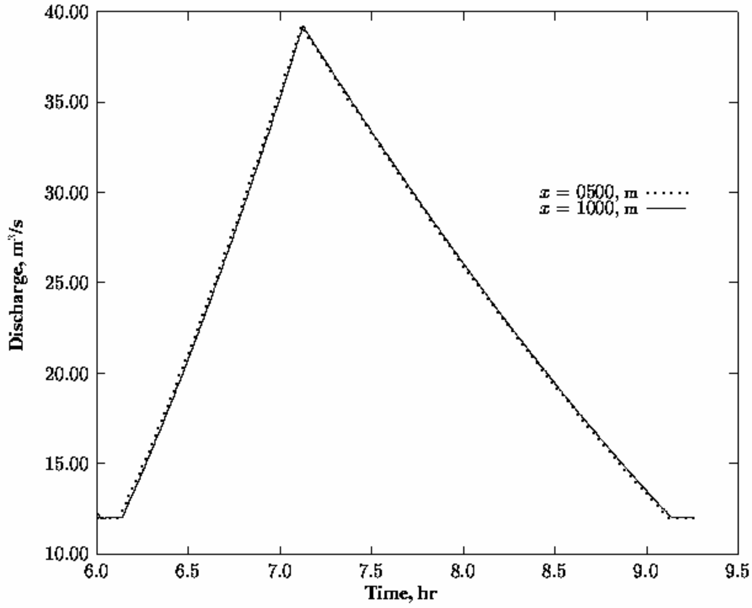


(a) Reach 1 at x=500 and x=1000 from the reservoir

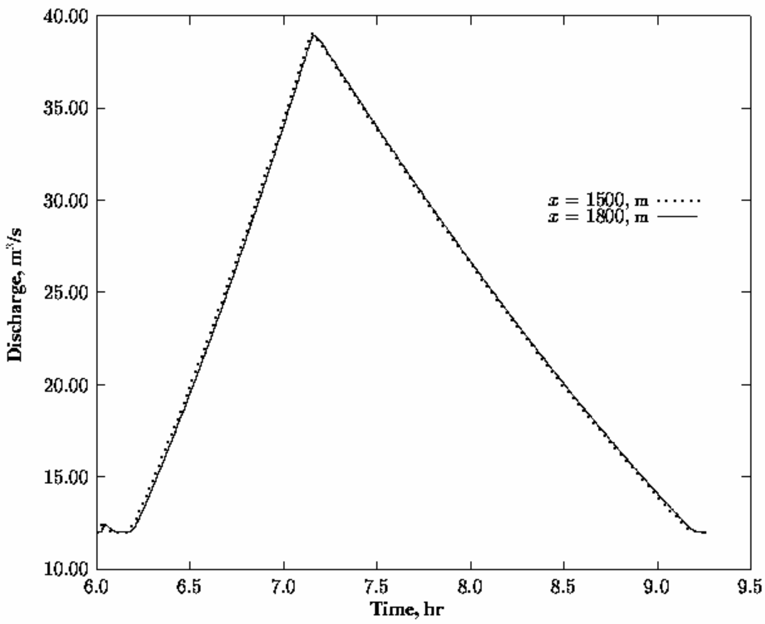


(b) Reach 3 at x=1500 and x=1800 from the reservoir

Fig. 7. Discharge versus time—MOC results



(a) Reach 1 at x=500 and x=1000 from the reservoir



(b) Reach 3 at x=1500 and x=1800 from the reservoir

Fig. 8. Discharge versus time—MIKE results

4. CONCLUSIONS

The suppression or reduction of the convective term in the numerical simulation of Saint-Venant equations may significantly affect the results in certain circumstances and should be considered carefully in trans-critical flow modeling while working with commercial software such as MIKE11. The salient point raised by this paper is the impact that convective term suppression might have on subsequent sub-critical flow downstream, which is the dramatic underestimation of water depth by as much as 20% in this region. Taking into account recent developments in the numerical modeling of supercritical and trans-critical flow, it may be better to gradually allow the users of professional software to have an option for the solution of non-reduced forms of governing equations.

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